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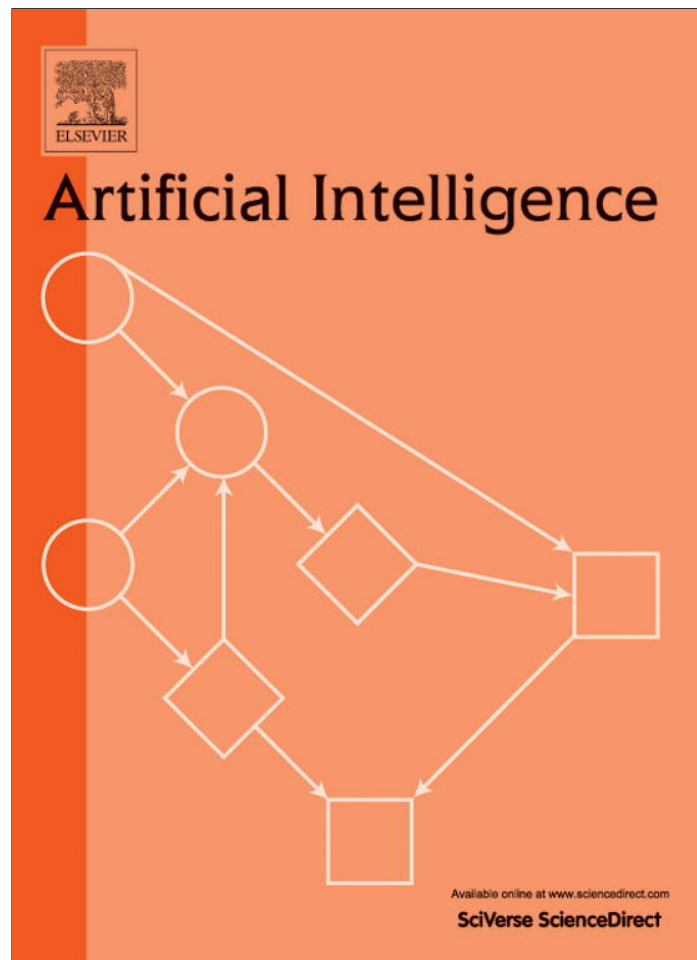
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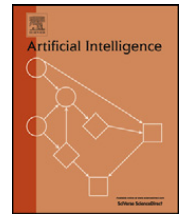
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On the succinctness of some modal logics



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ABSTRACT

One way of comparing knowledge representation formalisms that has attracted attention recently is in terms of *representational succinctness*, i.e., we can ask whether one of the formalisms allows for a more 'economical' encoding of information than the other. Proving that one logic is more succinct than another becomes harder when the underlying semantics is stronger. We propose to use *Formula Size Games* (as put forward by Adler and Immerman (2003) [1], but we present them as games for one player, called Spoiler), games that are played on two sets of models, and that directly link the *length* of a play in which Spoiler wins the game with the *size* of a formula, i.e., a formula that is true in the first set of models but false in all models of the second set. Using formula size games, we prove the following succinctness results for m -dimensional modal logic, where one has a set $I = \{i_1, \dots, i_m\}$ of indices for m modalities: (1) on general Kripke models (and also on binary trees), a definition $[\forall_\Gamma]\varphi = \bigwedge_{i \in \Gamma} [i]\varphi$ (with $\Gamma \subseteq I$) makes the resulting logic exponentially more succinct for $m > 1$; (2) several modal logics use such abbreviations $[\forall_\Gamma]\varphi$, e.g., in description logics the construct corresponds to adding role disjunctions, and an epistemic interpretation of it is 'everybody in Γ knows'. Indeed, we show that on epistemic models (i.e., S_5 -models), the logic with $[\forall_\Gamma]\varphi$ becomes more succinct for $m > 3$; (3) the results for the logic with 'everybody knows' also hold for a logic with 'somebody knows', and (4) on epistemic models, Public Announcement Logic is exponentially more succinct than epistemic logic, if $m > 3$. The latter settles an open problem raised by Lutz (2006) [18].

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1. Introduction

The study of the expressive power of logics is one of the major topics in mathematical logic and computer science. The general framework for such investigations can be described as follows. We begin with the question of whether a particular formalism can express some property on some class of models or not. The intuitive notion of property is given a formal expression through the concept of query and, therefore, the formal version of our initial question is whether a particular query is definable in some logic under investigation. Such questions are of great theoretical interest. However, it has been argued in [10] that, as far as knowledge representation formalisms are concerned, the comparison of two such formalisms, L_1 and L_2 , cannot be meaningfully accomplished just in terms of expressive power or the computational complexity of their inference problems. This is due to the fact that often we have the following situation:

1. L_1 and L_2 are equally expressive, and/or
2. L_1 and L_2 have the same complexity of the satisfiability problem, or

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3. The complexities of L_1 and L_2 are different but so high that it cannot be honestly claimed to be of any practical relevance.

Therefore, the authors of [10] suggest that a better comparison criterion is the *representational succinctness* of such formalisms. Intuitively, if we are interested in some particular query Q that is expressible with formulae φ_1 and φ_2 from L_1 and L_2 respectively, we can ask if there is a significant difference in the lengths of φ_1 and φ_2 . Hence, the notion of *succinctness* is a refinement of the notion of *expressivity*.

In this paper we present a number of succinctness results related to three well-known extensions of multimodal logic (ML) which have a popular epistemic and knowledge representation interpretation. A brief overview of our main theorems is as follows.

Adding formulae of the form $[\forall_\Gamma]\varphi$ to ML results in exponential succinctness on the well-known class of equivalence models \mathbf{S}_5 , the typical semantics for epistemic logic. Intuitively, a formula $[\forall_\Gamma]\varphi$ is best thought of as an abbreviation of the ML-formula $\bigwedge_{i \in \Gamma} [i]\varphi$. Such an abbreviation arises naturally in many branches of modal logic. For example, in epistemic logic [6,25], $[\forall_\Gamma]$ is called the ‘everybody knows’-modality. In boolean modal logic [9], $[\forall_\Gamma]$ corresponds to a modality of the form $[i_1 \cup \dots \cup i_n]$, where $\{i_1, \dots, i_n\} = \Gamma$; in the parlance of Description logics [2], $[\forall_\Gamma]$ corresponds to adding role disjunctions to the description logic \mathcal{ALC} (as in ‘sibling’ being defined as the role disjunction of ‘brother’ and ‘sister’). Finally, in dynamic logic [11], $[\forall_\Gamma]\varphi$ expresses that after every execution of any program from Γ , φ holds (demonic non-determinism).

Similarly, adding formulae of the form $[\exists_\Gamma]\varphi$ to ML results in exponential succinctness on \mathbf{S}_5 . A formula $[\exists_\Gamma]\varphi$ can be thought of as an abbreviation of the ML-formula $\bigvee_{i \in \Gamma} [i]\varphi$. Again, such formulae arise naturally in epistemic logic where the modality $[\exists_\Gamma]$ is called the ‘somebody knows’ modality. In Dynamic logic, this modality would represent angelic non-determinism: there is choice of a program from Γ , such that φ will hold after every execution of it.

Finally, adding formulae of the form $[\psi]\varphi$ to ML again results in exponential succinctness on \mathbf{S}_5 , which answers a question left open in [18]. The modal operator $[\psi]$ was introduced in [21] as a means for formalising the intuitive notion of ‘public announcement’. Intuitively, a formula $[\psi]\varphi$ is evaluated at a point w in a Kripke model by first discarding all points that do not satisfy ψ and then, if w has survived this procedure, we see whether φ is true at w in the newly obtained model.

The first of the above results can be explained in the following way. We show that for every natural number n , there is a set of \mathbf{S}_5 models \mathbb{M}^n and a property P of these models such that there is formula of the form $[\forall_\Gamma]\varphi$, whose length is linear in n , that expresses P but every equivalent formula from ML has length exponential in n . Similarly for the second and third results. This highlights the crucial importance of the class of models we use in our proofs. Intuitively, proving such a result with respect to a set of models \mathbb{N} for which we have no special requirements for the nature of the relations seems easier than when we impose additional conditions on the models. This is so, because the more conditions we impose on our models, the greater the chance to find a formula of sub-exponential length equivalent to $[\forall_\Gamma]\varphi$. Later we will see that such results depend not only on the class of models used but on the number of variables and relation symbols in the language, too.

The paper is organised as follows. In Section 2, we briefly introduce some classes of functions needed to define the notion of succinctness, or, better what it means that one logic is exponentially more succinct than another. We also provide a lemma (Lemma 1) which offers a sufficient condition to decide this: all our proofs of succinctness rely on this lemma. In this section, we also define the four modal languages ML, $[\forall_\Gamma]\text{ML}$, $[\exists_\Gamma]\text{ML}$ and $[\varphi]\text{ML}$ that we deal with in this paper. How do we demonstrate that any formula equivalent to $\psi \in L_2$ must have at least a certain length? In Section 2.3, we propose to use (an adaptation of) *Formula Size Games* (FSGs) introduced in [1]. FSGs establish a direct link between the number of moves needed for one player to win a game, and the length of formulae associated with the game (Theorem 1). We also prove the principle of diverging pairs (Theorem 2), which guarantees under which condition the number of moves needed to win certain sub-games, contribute to the number of moves to win the overall-game.

Then, Section 3 presents our succinctness results. In particular, in Section 3.1, we employ FSGs to show that both $[\forall_\Gamma]\text{ML}$ and $[\exists_\Gamma]\text{ML}$ are exponentially more succinct than ML, on the general class of Kripke models \mathbf{K} (Theorem 3). The theorem also establishes this for $[\varphi]\text{ML}$, but the proof for this is in [18]. Finally, Theorem 4 generalises this result of succinctness of these three modal languages to the class of models \mathbf{S}_5 , i.e., models where the underlying accessibility relations are equivalences. We conclude in Section 4, stating some open problems and conjectures.

The present paper is a greatly extended version of [8].

2. Preliminaries

2.1. Defining succinctness

When studying succinctness, we want to say that, in order to express a certain sequence of properties, the length of formulae in one language grows faster than in another language. To reason about relative growth of functions, one often uses either the so-called o-notation (also called asymptotic analysis), or a notation based on limits: both approaches are

equivalent (see e.g., [5]). We feel that the latter notation is often intuitively more clear, but on the other hand the o -notation enables us to prove Lemma 1. (The latter lemma provides a sufficient condition for proving succinctness: the reader interested in our succinctness results may start reading Lemma 1 and use that as a working definition of succinctness for the present paper.) For this reason, we present both notations, without here dwelling on their equivalence.

The sets of the natural, the positive natural and the positive real numbers are denoted \mathbb{N} , \mathbb{N}^+ , and \mathbb{R}^+ , respectively. All functions we consider are mappings from \mathbb{R}^+ to \mathbb{R}^+ (later, we will instantiate them as functions from \mathbb{N}^+ to \mathbb{N}^+ , as functions that measure growth of size of formulae in a sequence). We use the following standard notation (see for example [5]). Let $f(x)$, $g(x)$ denote functions of one variable, for which we will sometimes also write f and g . Unless stated otherwise, the variables c , z_0 and z range over \mathbb{R}^+ .

$$O(g(x)) = \{f(x) \mid \text{there are } c, z_0 \text{ such that for all } z \geq z_0: f(z) \leq cg(z)\} \quad (1.a)$$

$$o(g(x)) = \{f(x) \mid \text{for all } c \text{ there is } z_0 \text{ s.t. for all } z \geq z_0: f(z) < cg(z)\} \quad (1.b)$$

So, a function $f(x)$ is in $O(g(x))$ if there is a constant $c \in \mathbb{R}^+$, such that from some point $z_0 \in \mathbb{R}^+$ on, f is bounded by the constant times g . Using the alternative \lim -notation, we can rewrite (1.a) and (1.b) as (2.a) and (2.b), respectively:

$$f(x) \in O(g(x)) \text{ iff for some } c \in \mathbb{R}^+: \lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} \leq c \quad (2.a)$$

For f to be in $o(g(x))$ the requirement is much stronger: no matter how small the real number c is, after some point z_0 , the function f will be bound by c times g , i.e., be smaller. Equivalently:

$$f(x) \in o(g(x)) \text{ iff } \lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = 0 \quad (2.b)$$

As usual, the class $\text{poly}(x)$ of (single-variable) polynomial functions is defined as follows.

$$\text{poly}(x) = \bigcup_{c \in \mathbb{N}} O(x^c)$$

The class of exponential functions is denoted EXP and is identified with the class $2^{\text{poly}(x)}$. The class of sub-exponential functions, denoted SUBEXP , is identified with the class $2^{o(x)}$. That is,

$$h(x) \in \text{SUBEXP} \text{ iff } h(x) = 2^{f(x)} \text{ for some } f(x) \in o(x) \quad (1.c)$$

Using the alternative notation, we have $h(x) \in \text{SUBEXP}$ if $h(x) = 2^{f(x)}$ for some $f(x)$ with $\lim_{x \rightarrow \infty} \frac{f(x)}{x} = 0$, from which we derive:

$$h(x) \in \text{SUBEXP} \text{ iff } \lim_{x \rightarrow \infty} \frac{h(x)}{2^x} = 0 \quad (2.c)$$

The idea of SUBEXP is that it represents a class of functions that may grow faster than polynomials, yet not as fast as a 'proper' exponential function. Examples of functions in SUBEXP are polynomial functions, $\log(x)$, and $2^{\sqrt{x}}$.

Next, we define the notion of logic in a way that is sufficient for our purposes. For a more precise definition, the reader is invited to consult some of the standard textbooks on mathematical logic, e.g., [14].

Definition 1 (Logic). A logic $L = \langle \Phi, \models_L, \mathbb{M} \rangle$ is a triple where Φ is a non-empty set of formulae, \mathbb{M} is a non-empty class of models, and $\models_L \subseteq \mathbb{M} \times \Phi$ is a non-empty binary relation called truth relation. If the pair $(\mathcal{M}, \varphi) \in \models_L$, we write $\mathcal{M} \models_L \varphi$ and say that the formula φ is true in the model \mathcal{M} .

Note that we have not yet defined formulae, the truth relation \models_L , and the class of models \mathbb{M} . They are treated as parameters to be specified for the case at hand. Our only assumptions at this point are:

- Formulae are finite strings over a countable alphabet and the length of any formula φ , denoted $|\varphi|$, is the sum of the number of appearances of characters occurring in φ ;
- If $L_1 = \langle \Phi_1, \models_1, \mathbb{M}_1 \rangle$ and $L_2 = \langle \Phi_2, \models_2, \mathbb{M}_2 \rangle$ are two logics such that $\Phi_1 \subseteq \Phi_2$ and $\mathbb{M}_1 \subseteq \mathbb{M}_2$, then $\mathcal{M} \models_1 \varphi$ if and only if $\mathcal{M} \models_2 \varphi$ for any $\mathcal{M} \in \mathbb{M}_1$ and $\varphi \in \Phi_1$.

Of course, later, when we study specific logics, we will formally define formulae, the truth relation, the class of models, and formula length.

Definition 2 (Expressivity). Let $L_1 = \langle \Phi_1, \models_1, \mathbb{M} \rangle$ and $L_2 = \langle \Phi_2, \models_2, \mathbb{M} \rangle$ be two logics. We say that L_2 is at least as expressive as L_1 on the class of models \mathbb{M} , and write $L_1 \leq_{\mathbb{M}} L_2$, if and only if for every formula $\varphi_1 \in \Phi_1$, there is a formula $\varphi_2 \in \Phi_2$ such that for every $\mathcal{M} \in \mathbb{M}$, it is true that $\mathcal{M} \models_1 \varphi_1$ if and only if $\mathcal{M} \models_2 \varphi_2$. We say that the formula φ_2 is equivalent to φ_1 on \mathbb{M} , and write $\varphi_1 \equiv_{\mathbb{M}} \varphi_2$.

L_1 and L_2 are said to be equally expressive on \mathbb{M} , written $L_1 =_{\mathbb{M}} L_2$, if both $L_1 \leq_{\mathbb{M}} L_2$ and $L_2 \leq_{\mathbb{M}} L_1$ hold. As the reader would expect, $L_1 <_{\mathbb{M}} L_2$ is short for $L_1 \leq_{\mathbb{M}} L_2$ and $L_2 \not\leq_{\mathbb{M}} L_1$

The next definition, first given in [12] and [13], introduces the notion of succinctness as a refinement of expressivity.

Definition 3 (*Succinctness*). Let $L_1 = \langle \Phi_1, \models_1, \mathbb{M} \rangle$ and $L_2 = \langle \Phi_2, \models_2, \mathbb{M} \rangle$ be two logics such that $L_1 \leq_{\mathbb{M}} L_2$. Let F be a class of functions.

- We say that L_1 is F -succinct in L_2 on \mathbb{M} , and write $L_1 \leq_{\mathbb{M}}^F L_2$, if and only if there is a function $f \in F$ such that for every $\varphi_1 \in \Phi_1$ there is a formula $\varphi_2 \in \Phi_2$ which is equivalent to φ_1 on \mathbb{M} such that $|\varphi_2| \leq f(|\varphi_1|)$. We write $L_1 \not\leq_{\mathbb{M}}^F L_2$ if it is not the case that L_1 is F -succinct in L_2 on \mathbb{M} .
- We say that L_1 is exponentially more succinct than L_2 on \mathbb{M} if and only if $L_1 \not\leq_{\mathbb{M}}^{\text{SUBEXP}} L_2$.

Intuitively, when we say that L_1 is F -succinct in L_2 on \mathbb{M} , not only do we mean that L_2 is at least as expressive as L_1 on \mathbb{M} , but, in addition, we can give an F -upper bound on the size of L_2 -formulae needed to express all of L_1 on \mathbb{M} . However, if the length of the L_2 -formulae expressing all of L_1 on \mathbb{M} cannot be bounded from above by a sub-exponential function, we say that L_1 is exponentially more succinct than L_2 on \mathbb{M} . We would like to stress that $L_1 \not\leq_{\mathbb{M}}^F L_2$ and $L_2 \not\leq_{\mathbb{M}}^F L_1$ can be both true at the same time (see for example [24] for one such result). Likewise, it is possible for three languages that L_1 is exponentially more succinct than L_2 which in turn is exponentially more succinct than L_3 , without L_1 being exponentially more succinct than L_3 . This is the case when some properties are more economically expressed in L_1 than in L_2 , and some (other) properties are more economically expressed in L_2 than in L_3 .

The simple proposition below follows immediately from Definitions 1, 2, 3, and our assumption about the properties of the relation \models .

Proposition 1. Let $L_1 = \langle \Phi_1, \models_1, \mathbb{K} \rangle$, $L_2 = \langle \Phi_2, \models_2, \mathbb{K} \rangle$, and $L_3 = \langle \Phi_3, \models_3, \mathbb{M} \rangle$, $L_4 = \langle \Phi_4, \models_4, \mathbb{M} \rangle$ be four logics for which the following are true:

- $L_1 \not\leq_{\mathbb{K}}^F L_2$;
- $\Phi_1 \subseteq \Phi_3$;
- $\Phi_2 = \Phi_4$;
- $\mathbb{K} \subseteq \mathbb{M}$.

Then $L_3 \not\leq_{\mathbb{M}}^F L_4$.

It follows in particular from Proposition 1, in the case that $\Phi_1 = \Phi_3$, and $F = \text{SUBEXP}$, that if a logic L_1 is exponentially more succinct than L_2 on a restricted class of models \mathbb{K} , the same language L_1 is also exponentially more succinct than L_2 on a class of models \mathbb{M} such that $\mathbb{K} \subseteq \mathbb{M}$.

Next, we state a lemma that provides us with a sufficient condition for proving that a logic L_1 is exponentially more succinct than another logic L_2 .

Lemma 1. Let $L_1 = \langle \Phi_1, \models_1, \mathbb{M} \rangle$ and $L_2 = \langle \Phi_2, \models_2, \mathbb{M} \rangle$ be two logics such that $L_1 \leq_{\mathbb{M}} L_2$ and let $f(x) \in O(g(x))$ be a strictly increasing function. Suppose that for every $n \in \mathbb{N}$, there are two formulae $\alpha_n \in \Phi_1$ and $\beta_n \in \Phi_2$ satisfying the properties:

1. $|\alpha_n| = f(n)$;
2. $|\beta_n| \geq 2^{g(n)}$;
3. β_n is the shortest formula in Φ_2 that is equivalent to α_n on \mathbb{M} .

Then $L_1 \not\leq_{\mathbb{M}}^{\text{SUBEXP}} L_2$.

Proof. We have to prove that for every function $i(x) \in \text{SUBEXP}$ there is a formula $\varphi_i \in \Phi_1$ such that $|\theta| > i(|\varphi_i|)$ for any formula $\theta \in \Phi_2$ that is equivalent to φ_i on \mathbb{M} .

Since $f(x) \in O(g(x))$, there are natural numbers c and z_0 such that $f(z) \leq cg(z)$ for all $z \geq z_0$. Hence $\frac{1}{c}f(z) \leq g(z)$ for all $z \geq z_0$. Suppose now that $i(x) \in \text{SUBEXP}$. Then there is a natural number m_0 such that $i(m) < 2^{\frac{1}{c}m}$ for all $m \geq m_0$. Since $f(x)$ is strictly increasing, there is a natural number $\mathbf{i} \geq n_0$, such that $|\alpha_{\mathbf{i}}| = f(\mathbf{i}) \geq m_0$. Therefore, $i(|\alpha_{\mathbf{i}}|) < 2^{\frac{1}{c}|\alpha_{\mathbf{i}}|}$. Now, the statement follows from the condition that $\beta_{\mathbf{i}}$ is the shortest formula that is equivalent to $\alpha_{\mathbf{i}}$, the fact that $|\beta_{\mathbf{i}}| \geq 2^{g(\mathbf{i})}$, and $g(\mathbf{i}) \geq \frac{1}{c}f(\mathbf{i}) = \frac{1}{c}|\alpha_{\mathbf{i}}|$. \square

Corollary 1. Suppose the conditions of Lemma 1 are met. Then there are infinitely many pairwise non-equivalent formulae on \mathbb{M} among the formulae α_n , $n \geq 1$.

Proof. Suppose that there are only finitely many non-equivalent formulae on \mathbb{M} $\alpha_i, \dots, \alpha_k$ among the formulae $\alpha_1, \dots, \alpha_n, \dots$. Let β_i, \dots, β_k be the respective equivalent formulae on \mathbb{M} in the sequence $\beta_1, \dots, \beta_n, \dots$. Since any other

formula α_j , where $j \notin \{i, \dots, k\}$ in the sequence $\alpha_1, \dots, \alpha_n, \dots$ is equivalent on \mathbb{M} to one of the formulae $\alpha_i, \dots, \alpha_k$, it follows that every α_l formula is equivalent on \mathbb{M} to one of the formulae β_i, \dots, β_k . Therefore, for any sufficiently large index l , it is not true that β_l is the shortest formula that is equivalent on \mathbb{M} to α_l and at the same time $|\beta_l| \geq 2^l$. \square

We finish this subsection with some comments on Lemma 1 and Definition 3.

Note that the condition $f(x) \in O(g(x))$ in the statement of the lemma is essential and cannot be replaced by $g(x) \in O(f(x))$. To see this, consider the following example:

Example 1. Suppose that for every $n \in \mathbb{N}$, there are two formulae $\alpha_n \in \Phi_1$ and $\beta_n \in \Phi_2$ satisfying the properties:

- 1'. $|\alpha_n| = n^3$;
- 2'. $|\beta_n| = 2^n$;
- 3'. β_n is the shortest formula in Φ_2 that is equivalent to α_n on \mathbb{K} .

It is obvious that $n \in O(n^3)$. In this case however, we cannot use these two sequences of formulae to prove that there is an exponential succinctness gap between the logics L_1 and L_2 because the sub-exponential function $2^{\sqrt[3]{x}}$ bounds the length of each β_n in the length of α_n , i.e., $|\beta_n| = 2^{\sqrt[3]{|\alpha_n|}}$. Hence, we must impose a stronger condition on the length of $|\beta_n|$, namely, $|\beta_n| = 2^{f(n)}$ where $n^3 \in O(f(n))$.

To the best of our knowledge, [12] and [13] are the first papers that gave an explicit general definition of the notion of one logic *being exponentially more succinct* than another. Some articles (e.g. [16,18–20,22]) use instances of Lemma 1 implicitly, while others define explicitly ‘exponentially more succinct’ as some particular instance of Lemma 1. For example, the following definition is used in [23].

If two languages, L_1 and L_2 , are equally expressive, we say that L_1 is exponentially more succinct than L_2 if there is an infinite sequence of L_1 expressions R_1, R_2, \dots where the length of R_k is polynomial in k , such that for every sequence of equivalent L_2 -expressions P_1, P_2, \dots , the length of P_k is exponential in k .

Undoubtedly, such a definition provides a sufficient condition¹ for L_1 to be exponentially more succinct than L_2 , but, unfortunately, it does not fit results like the one in, e.g., [22], where, in the wording above, the length of the expression R_k is logarithmic in k while the length of P_k is linear in k . Of course this case is covered by Lemma 1.

2.2. Multimodal logic

In this subsection, we define the logics that we study in the rest of the paper.

Definition 4 (Formulae). A signature is a pair $\Sigma = \langle A, I \rangle$, where $A = \{p_1, p_2, \dots\}$ is a countable set of propositional (or ‘atomic’) symbols and $I = \{i_1, i_2, \dots, i_n\}$ is a finite set² of indices. Let p, i and Γ vary over A, I , and the non-empty subsets of I , respectively. The formulae of the Multimodal Logic ML and its extensions $[\forall_\Gamma]\text{ML}$, $[\exists_\Gamma]\text{ML}$, and $[\varphi]\text{ML}$ in the signature $\Sigma = \langle A, I \rangle$ are built as follows.

- The set Φ_{ML} of formulae of Multimodal Logic ML consists of all strings ψ constructed according to the rule:

$$\varphi := p \mid \neg\varphi \mid (\varphi \vee \varphi) \mid [i]\varphi;$$

- The set of formulae $\Phi_{[\forall_\Gamma]\text{ML}}$ of the logic $[\forall_\Gamma]\text{ML}$ consists of the strings:

$$\varphi := p \mid \neg\varphi \mid (\varphi \vee \varphi) \mid [i]\varphi \mid [\forall_\Gamma]\varphi;$$

- The set $\Phi_{[\exists_\Gamma]\text{ML}}$ of formulae of the logic $[\exists_\Gamma]\text{ML}$, consists of the strings:

$$\varphi := p \mid \neg\varphi \mid (\varphi \vee \varphi) \mid [i]\varphi \mid [\exists_\Gamma]\varphi;$$

- The set $\Phi_{[\varphi]\text{ML}}$ of formulae of the logic $[\varphi]\text{ML}$ consists of the strings:

$$\varphi := p \mid \neg\varphi \mid (\varphi \vee \varphi) \mid [i]\varphi \mid [\varphi]\varphi;$$

¹ Provided that we are careful what ‘exponential in k ’ means in this case as explained in Example 1.

² We follow a common convention in modal logic that the number of modalities, or indices is finite. Allowing for infinite sets of indices would not negatively affect our main results, however. In that case, we would have two options: (1) while allowing I to be infinite, we would require that the subsets Γ that occur in the formulas must be finite. In that case, we would still have $\text{ML} =_{\mathbf{K}} [\forall_\Gamma]\text{ML}$ and $\text{ML} =_{\mathbf{K}} [\exists_\Gamma]\text{ML}$. Alternatively, (2) we also allow the Γ ’s that occur in formulas to be infinite. In that case, we would have $\text{ML} \leq_{\mathbf{K}} [\forall_\Gamma]\text{ML}$ and $\text{ML} \leq_{\mathbf{K}} [\exists_\Gamma]\text{ML}$ (rather than $=_{\mathbf{K}}$ in both cases), a condition that is a pre-requisite for succinctness (see Definition 3). Under the second alternative, we would also need to specify that for instance $[[\forall_\Gamma]\varphi]$ is infinite whenever Γ is.

We will also refer to sets of formulae as just defined as *languages*.

As suggested in the introduction, there are plenty of examples of such languages. Let us here restrict ourselves to an epistemic or doxastic setting. Then, I represents a set of agents, $[i]\varphi$ (which would be typically written $K_i\varphi$) would denote that agent i knows φ , in a doxastic setting $[i]\varphi$ ($B_i\varphi$) would express that i believes that φ . In an epistemic context, the formulae of the form $[\varphi]\psi$ are to be read as ‘after a public announcement that φ , property ψ holds’. Likewise, in the same context, $[\forall_I]\varphi$ (typically written $E_I\varphi$) means that everybody in I knows φ , whereas $[\exists_I]\varphi$ denotes that somebody in I knows that φ .

Definition 5 (*Length of formulae*). The length of a formula φ is denoted $|\varphi|$. It is defined as follows: $|p| = 1$, $|(\varphi_1 \vee \varphi_2)| = |\varphi_1| + |\varphi_2| + 1$, $|\neg\varphi| = |[i]\varphi| = |[\exists_I]\varphi| = |[\forall_I]\varphi| = 1 + |\varphi|$, and $|[\varphi_1]\varphi_2| = |\varphi_1| + |\varphi_2|$.

As usual, we have the following abbreviations $\top = p \vee \neg p$, $\perp = \neg\top$, $(\varphi \wedge \psi) = \neg(\neg\varphi \vee \neg\psi)$, $\langle i \rangle\varphi = \neg[i]\neg\varphi$, and $\langle \varphi \rangle\psi = \neg[\varphi]\neg\psi$. It is important to note though that by defining the languages as we did, succinctness results are given in the most general way, in the following sense.

Definition 6. Let Φ be a set of formulae, and $L = \langle \Phi, \models, \mathbb{M} \rangle$ a logic. We say that Φ' adds linear definitions in L to Φ if there is a mapping $\tau : \Phi' \rightarrow \Phi$ such that

1. $\tau(\alpha)$ is equivalent to α on \mathbb{M} for all $\alpha \in \Phi$;
2. $\exists k \in \mathbb{N} \forall \alpha \in \Phi \quad |\tau(\alpha)| \leq k|\alpha|$.

The smallest k for which the second item holds is called *the scalar* for Φ' (with respect to Φ).

As an example, take Φ to be any of the sets of formulae defined above, and $L = \langle \Phi, \models, \mathbf{K} \rangle$, the set of formulae Φ' obtained from Φ by adding the definitions of $\varphi \wedge \psi$, \perp and $\langle i \rangle\varphi$. Then Φ' adds linear definitions in L to Φ . In this case, $k \geq 5$. Note that if $|\tau(\alpha)| \leq 5|\alpha|$ and $|\tau(\beta)| \leq 5|\beta|$, we have $|\tau(\alpha \wedge \beta)| = |\neg(\neg\tau(\alpha) \vee \neg\tau(\beta))| \leq 5 + |\tau(\alpha)| + |\tau(\beta)| \leq 5 + 5|\alpha| + 5|\beta| = 5|\alpha \wedge \beta|$, while $|\tau(\perp)| = |\neg(\neg p \vee p)| = 5 \leq 5|\perp|$.

Proposition 2. Let Φ_1, Φ_2 be any of the languages defined in Definition 4, and let $L_1 = \langle \Phi_1, \models_1, \mathbb{M} \rangle$ and $L_2 = \langle \Phi_2, \models_2, \mathbb{M} \rangle$ be two logics. Suppose Φ_i adds linear definitions in L_i with scalar k_i ($i = 1, 2$). Let $L'_i = \langle \Phi'_i, \models_i, \mathbb{M} \rangle$ ($i = 1, 2$). Then:

L_1 is exponentially more succinct than L_2 on \mathbb{M} iff

L'_1 is exponentially more succinct than L'_2 on \mathbb{M}

Proof. Let $k = \max\{k_1, k_2\}$. Then, for $i = 1, 2$, we have that for every $\varphi'_i \in \Phi'_i$ there is $\varphi_i \in \Phi_i$, such that φ_i is equivalent to φ'_i on \mathbb{M} and $|\varphi_i| \leq k|\varphi'_i|$. Now, recall that L_1 is exponentially more succinct than L_2 on \mathbb{M} iff

$$\forall f \in SUBEXP \quad \exists \varphi_1 \in \Phi_1 \quad \forall \varphi_2 \in \Phi_2 \quad \varphi_1 \equiv_{\mathbb{M}} \varphi_2 \Rightarrow |\varphi_2| > f(|\varphi_1|) \quad (1)$$

Assume (1) holds and take an arbitrary $g \in SUBEXP$. Define $f(x) = kg(x)$. Then $f \in SUBEXP$ and hence $\exists \varphi_1 \in \Phi_1 \subseteq \Phi'_1$ such that $\forall \varphi_2 \in \Phi_2$ we have $\varphi_1 \equiv_{\mathbb{M}} \varphi_2 \Rightarrow |\varphi_2| > f(|\varphi_1|)$. Now fix this φ_1 and choose $\varphi'_2 \in \Phi'_2$ such that $\varphi'_2 \equiv_{\mathbb{M}} \varphi_1$ arbitrarily. We have $|\varphi'_2| \geq \frac{1}{k}|\varphi_2| > \frac{1}{k}f(|\varphi_1|) = g(|\varphi_1|)$, which proves that L'_1 is exponentially more succinct than L'_2 on \mathbb{M} .

For the other direction, suppose

$$\forall g \in SUBEXP \quad \exists \varphi'_1 \in \Phi'_1 \quad \forall \varphi'_2 \in \Phi'_2 \quad \varphi'_1 \equiv_{\mathbb{M}} \varphi'_2 \Rightarrow |\varphi'_2| > g(|\varphi'_1|) \quad (2)$$

Take an arbitrary $f \in SUBEXP$ and define $f_1(x) = \max\{f(y) \mid y \leq x\}$. Then we have that $f_1(x)$ is a non-decreasing function³ in $SUBEXP$ such that $f \leq f_1$. Next, define $f_2(x) = f_1(kx)$. Again, $f_2 \in SUBEXP$ and $f_2 \geq f_1$. By (2), we have that $\exists \varphi'_1 \in \Phi'_1$ such that $\forall \varphi'_2 \in \Phi'_2$, the implication $\varphi'_1 \equiv_{\mathbb{M}} \varphi'_2 \Rightarrow |\varphi'_2| > f_2(|\varphi'_1|)$ holds. Take this $\varphi'_1 \in \Phi'_1$ and consider its equivalent $\varphi_1 \in \Phi_1$ with $k|\varphi'_1| \geq |\varphi_1|$. Take an arbitrary $\varphi_2 \in \Phi_2 \subseteq \Phi'_2$ that is equivalent on \mathbb{M} to φ_1 . We then have

$$|\varphi_2| > f_2(|\varphi'_1|) \geq f_1(|\varphi_1|) \geq f(|\varphi_1|)$$

Since this holds for arbitrary $f \in SUBEXP$, this shows that L_1 is exponentially more succinct than L_2 on \mathbb{M} . \square

The class of models for the logics of Definition 4 and the respective truth relations are defined in the usual way (e.g., see [3]), i.e, using Kripke models.

³ I.e., $\forall x, y \in \mathbb{R}^+ (x \geq y \Rightarrow f_1(x) \geq f_1(y))$.

Definition 7 (Kripke model). A Kripke model for the signature $\Sigma = \langle A, I \rangle$ is a triple $\mathcal{M} = \langle M, R, V \rangle$, where M is a non-empty set, the members of which we will refer to as *points*, $R : I \rightarrow 2^{M \times M}$ is a mapping that assigns a binary relation on M to every $i \in I$, and $V : A \rightarrow 2^M$ is a function that assigns a subset of M to every $p \in A$. We write $sR_i t$ for $(s, t) \in R(i)$ and say that t is an i -successor of s or that s and t are i -connected, or connected with an edge labelled i .

The model $\mathcal{M} = \langle M, R, V \rangle$ is said to be *finite* if M is finite. \mathcal{M} is called *image-finite* if for every $w \in M$ and every $R(i)$, the set $\{v \mid wR_i v\}$ is finite. The pair (\mathcal{M}, w) , where $w \in M$, is called *pointed model*. Sets of pointed models are denoted $\mathbb{A}, \mathbb{B}, \dots$. The class of all pointed models is denoted \mathbf{K} . Other important classes of models in this paper are **KD45** and **S5**: they provide the most popular models for languages that interpret $[i]$ as belief and knowledge of an agent i , respectively. Models $\mathcal{M} = \langle M, R, V \rangle$ in **KD45** are such that every $R(i)$ is serial (i.e., $\forall x \in M \exists y \in M xR_i y$), transitive and Euclidean ($\forall xyz((xR_i y \ \& \ xR_i z) \Rightarrow yR_i z)$). Finally, in models $\mathcal{M} = \langle M, R, V \rangle$ in **S5**, every $R(i)$ is an equivalence relation.

Definition 8 (Truth). Let $(\mathcal{M}, w) \in \mathbf{K}$ and φ be a formula. The relation “ φ is true in the pointed model (\mathcal{M}, w) ”, written $(\mathcal{M}, w) \models \varphi$, is defined inductively on the structure of φ as follows.

- $(\mathcal{M}, w) \models p$ if and only if $w \in V(p)$;
- $(\mathcal{M}, w) \models \neg\psi$ if and only if $\text{not } (\mathcal{M}, w) \models \psi$;
- $(\mathcal{M}, w) \models \psi_1 \vee \psi_2$ if and only if $(\mathcal{M}, w) \models \psi_1$ or $(\mathcal{M}, w) \models \psi_2$;
- $(\mathcal{M}, w) \models [i]\psi$ if and only if for all v , $wR_i v$ implies $(\mathcal{M}, v) \models \psi$;
- $(\mathcal{M}, w) \models [\exists_\Gamma]\psi$ if and only if there is an $i \in \Gamma$ such that $(\mathcal{M}, w) \models [i]\psi$;
- $(\mathcal{M}, w) \models [\forall_\Gamma]\psi$ if and only if for all $i \in \Gamma$, $(\mathcal{M}, w) \models [i]\psi$;
- $(\mathcal{M}, w) \models [\psi_1]\psi_2$ if and only if If $(\mathcal{M}, w) \models \psi_1$, then $(\mathcal{M}|_{\psi_1}, w) \models \psi_2$.

Intuitively, the model $\mathcal{M}|_{\psi_1}$ used to define the \models relation for the formula $[\psi_1]\psi_2$ is the restriction of the model \mathcal{M} to the points in which ψ_1 is true. Formally, for any formula φ , and any model $\mathcal{M} = \langle M, R, V \rangle$, the model $\mathcal{M}|_\varphi = \langle M', R', V' \rangle$, is such that $M' = \{v \in M \mid (\mathcal{M}, v) \models \varphi\}$, and R' and V' are the restrictions of R and V to M' . We will, in Section 3, also use the dual $\langle \varphi \rangle$ of $[\varphi]$, where $\langle \varphi \rangle$ is defined as $\neg[\varphi]\neg\psi$: in other words, $(\mathcal{M}, w) \models \langle \varphi \rangle\psi$ if and only if $(\mathcal{M}, w) \models \varphi$ and $(\mathcal{M}|_\varphi, w) \models \psi$. In words: it is possible to announce φ (which in turn means that φ holds), and after announcing it, ψ holds. Some properties of $\langle \varphi \rangle$ are given in the proof of Proposition 5.

If \mathbb{A} is a set of pointed models and φ is a formula of one of the logics above, we write $\mathbb{A} \models \varphi$ to mean that for all $(\mathcal{M}, w) \in \mathbb{A}$, $(\mathcal{M}, w) \models \varphi$.

We were not completely precise in defining the truth relation \models , because, technically speaking, we have to specify four truth relations corresponding to the four logics ML, $[\forall_\Gamma]\text{ML}$, $[\exists_\Gamma]\text{ML}$, and $[\varphi]\text{ML}$. Such precision, however, will unnecessarily complicate our exposition; moreover, it will always be clear from the context which of the truth relations we mean.

2.3. Formula size games

Formula Size Games or Adler–Immerman games were introduced in the seminal [1] as a generalisation of Ehrenfeucht–Fraïssé games that enable us to reason not only about the quantifier depth of a first-order formula but about its length, too. The versatility of these games stems from the fact that we can formulate a suitable version for practically any logic. However, it was noted already in [1] that unlike Ehrenfeucht–Fraïssé games, they are not truly two-player games because the second player, has in a sense an “optimal” answer to every move by the first player (the player we call Spoiler). This optimal answer can be incorporated in the definition of the game and, therefore, there is no need for a second player. What is more, these games can be replaced completely by the so-called extended syntax trees defined in the important [13].

Here, we will adopt the middle ground position and define a suitable one-player version. We opted to keep the intuitively appealing games and give complete proofs of the properties we need. For another application of formula size games to obtaining lower bounds on the size of modal logic formulae, the reader is invited to consult [7].

Definition 9 (Formula size games). The one-person (called Spoiler) formula size game (FSG) on two sets of pointed models \mathbb{A} and \mathbb{B} is played as follows. During the course of the game, a game tree is constructed in such a way that each node is labelled with a pair $\langle \mathbb{C}, \mathbb{D} \rangle$ of sets of pointed models and one symbol from the set $\Sigma = \{p, \neg, \vee, [i]\}$. A node labelled with the pair $\langle \mathbb{C}, \mathbb{D} \rangle$ is denoted $\langle \mathbb{C} \circ \mathbb{D} \rangle$. The models in \mathbb{C} are called the models on the left. Similarly, the models in \mathbb{D} are called the models on the right.

A node in the tree can be declared either *open* or *closed*. Once a node has been declared “closed”, no further game-moves can be played at it. Moves can be played only at open nodes. The game begins with the root of the game tree $\langle \mathbb{A} \circ \mathbb{B} \rangle$ that is declared “open”.

Let an open node $\langle \mathbb{C} \circ \mathbb{D} \rangle$ be given. Spoiler can make one of the following moves at this node:

atomic-move: Spoiler chooses a propositional symbol p such that $\mathbb{C} \models p$ and $\mathbb{D} \models \neg p$. The node is declared closed and labelled with the symbol p .

not-move: Spoiler labels the node with the symbol \neg and adds one new open node $\langle \mathbb{D} \circ \mathbb{C} \rangle$ as a successor to the node $\langle \mathbb{C} \circ \mathbb{D} \rangle$.

or-move: Spoiler labels the node with the symbol \vee and chooses two subsets $\mathbb{C}_1 \subseteq \mathbb{C}$ and $\mathbb{C}_2 \subseteq \mathbb{C}$ such that $\mathbb{C} = \mathbb{C}_1 \cup \mathbb{C}_2$. Two new open nodes are added to the tree as successors to the node $\langle \mathbb{C} \circ \mathbb{D} \rangle$, namely $\langle \mathbb{C}_1 \circ \mathbb{D} \rangle$ and $\langle \mathbb{C}_2 \circ \mathbb{D} \rangle$.

[i]-move: Spoiler labels the node with the symbol $[i]$ and, for each pointed model $(\mathcal{S}, v) \in \mathbb{D}$, he chooses a pointed model (\mathcal{S}, v') such that $v R_i v'$ (if for some $(\mathcal{S}, v) \in \mathbb{D}$ this is not possible, Spoiler cannot play this move). All these new pointed models are collected in the set \mathbb{D}_1 . A set of models \mathbb{C}_1 is then constructed as follows. For each pointed model $(\mathcal{C}, w) \in \mathbb{C}$, all the possible pointed models (\mathcal{C}, w') such that $w R_i w'$ are added to \mathbb{C}_1 . If for some (\mathcal{C}, w) , the point w does not have an R_i -successor, nothing is added to \mathbb{C}_1 for the pointed model (\mathcal{C}, w) . A new open node $\langle \mathbb{C}_1 \circ \mathbb{D}_1 \rangle$ is added as a successor to the node $\langle \mathbb{C} \circ \mathbb{D} \rangle$. In this case, we also say that Spoiler has played an index-move.

Definition 10 (*Winning condition for FSG*). We say that Spoiler wins the FSG starting at $\langle \mathbb{A} \circ \mathbb{B} \rangle$ in n moves if and only if there is a game tree T with root $\langle \mathbb{A} \circ \mathbb{B} \rangle$ and precisely n nodes such that every leaf of T is closed.

The next theorem connects the formula size games with the length of formulae of ML.

Theorem 1. Spoiler can win the FSG starting at $\langle \mathbb{A} \circ \mathbb{B} \rangle$ in less than k moves if and only if there is some $n < k$ and a formula $\varphi \in \Phi_{\text{ML}}$ such that $\mathbb{A} \models \varphi$, $\mathbb{B} \models \neg\varphi$, and $|\varphi| = n$.

Proof. (If) Suppose that there is a formula φ of size $n < k$ such that $\mathbb{A} \models \varphi$ and $\mathbb{B} \models \neg\varphi$. We prove by induction on the structure of φ that Spoiler can win the game starting in $\langle \mathbb{A} \circ \mathbb{B} \rangle$ in n moves by playing according to φ .

Base case

If φ is the propositional variable p , then Spoiler plays the atomic-move and the tree is closed, as required. It is obvious that the tree has just one node, i.e., Spoiler can win the game in $|\varphi|$ moves by playing according to φ .

Induction step

- If φ is $\neg\psi$, then Spoiler plays the *not*-move by adding the node $\langle \mathbb{B} \circ \mathbb{A} \rangle$ as a successor to $\langle \mathbb{A} \circ \mathbb{B} \rangle$. Clearly, $\mathbb{B} \models \psi$, $\mathbb{A} \models \neg\psi$, and $|\psi| = n - 1$. Applying the induction hypothesis, Spoiler can win the sub-game starting at $\langle \mathbb{B} \circ \mathbb{A} \rangle$ in $n - 1$ moves by playing according to ψ , hence Spoiler can win in n moves the game starting at $\langle \mathbb{A} \circ \mathbb{B} \rangle$ by playing according to $\neg\psi$.
- If φ is $\psi_1 \vee \psi_2$, then Spoiler plays the *or*-move and adds two new leaves to the tree, namely $\langle \mathbb{A}_1 \circ \mathbb{B} \rangle$ and $\langle \mathbb{A}_2 \circ \mathbb{B} \rangle$, where $\mathbb{A} = \mathbb{A}_1 \cup \mathbb{A}_2$, $\mathbb{A}_1 \models \psi_1$ and $\mathbb{A}_2 \models \psi_2$. Applying the induction hypothesis, we see that Spoiler can win in $|\psi_1|$ moves the sub-game starting at $\langle \mathbb{A}_1 \circ \mathbb{B} \rangle$, and he can win in $|\psi_2|$ moves the sub-game starting at $\langle \mathbb{A}_2 \circ \mathbb{B} \rangle$. Therefore, he can win the game starting at $\langle \mathbb{A} \circ \mathbb{B} \rangle$ in $|\psi_1 \vee \psi_2| = |\psi_1| + |\psi_2| + 1 = n$ moves by playing according to φ .
- If φ is a formula of the form $[i]\psi$, then for each model $(\mathcal{B}, v) \in \mathbb{B}$, Spoiler chooses a model (\mathcal{B}, v') such that $v R_i v'$ and $(\mathcal{B}, v') \models \neg\psi$. Let \mathbb{B}_1 be the set of models Spoiler has chosen. For each pointed model $(\mathcal{A}, w) \in \mathbb{A}$, all possible pointed models (\mathcal{A}, w') such that $w R_i w'$ are collected in the set \mathbb{A}_1 . A new node $\langle \mathbb{A}_1 \circ \mathbb{B}_1 \rangle$ is added as a successor of $\langle \mathbb{A} \circ \mathbb{B} \rangle$. Clearly, $\mathbb{A}_1 \models \psi$ and $\mathbb{B}_1 \models \neg\psi$. Applying the induction hypothesis, we see that Spoiler can win the sub-game starting at $\langle \mathbb{A}_1 \circ \mathbb{B}_1 \rangle$ in $|\psi|$ moves. Therefore, Spoiler can win the FSG starting at $\langle \mathbb{A} \circ \mathbb{B} \rangle$ in $|\varphi| = |[i]\psi| = |\psi| + 1 = n$ moves by playing according to φ .

(Only if) Suppose that Spoiler has won the formula size game starting at $\langle \mathbb{A} \circ \mathbb{B} \rangle$ in $n < k$ moves. We claim that the resulting closed game tree is a parse tree of a formula φ of length n such that $\mathbb{A} \models \varphi$ and $\mathbb{B} \models \neg\varphi$ (in such a case, we will also say that Spoiler plays ‘according to φ ’). In order to prove this, we label the nodes of the tree step by step with formulae, starting with the leaves. These were labelled during the game with the propositional variables p that Spoiler used to close them. Then the rest of the nodes are labelled successively. If a node has a \neg label and its successor is labelled with ψ , then that node is labelled with $\neg\psi$. If a node has an \vee label and its two successors are labelled with ψ and χ , then that node is labelled with $\psi \vee \chi$, respectively. If a node has a $[i]$ label and its successor is labelled with ψ , then that node is labelled with $[i]\psi$.

By a straightforward backward induction on the tree we can see that for each node $\langle \mathbb{C} \circ \mathbb{D} \rangle$, the following are true.

- The string of symbols labelling the node is indeed a formula of ML.
- The formula labelling the node is true in all the models in \mathbb{C} and false in all the models in \mathbb{D} . Therefore, the formula labelling the root of the tree is true in all the pointed models in \mathbb{A} and false in all the pointed models in \mathbb{B} .

It is obvious that the game tree is a parse tree for the formula labelling the root. \square

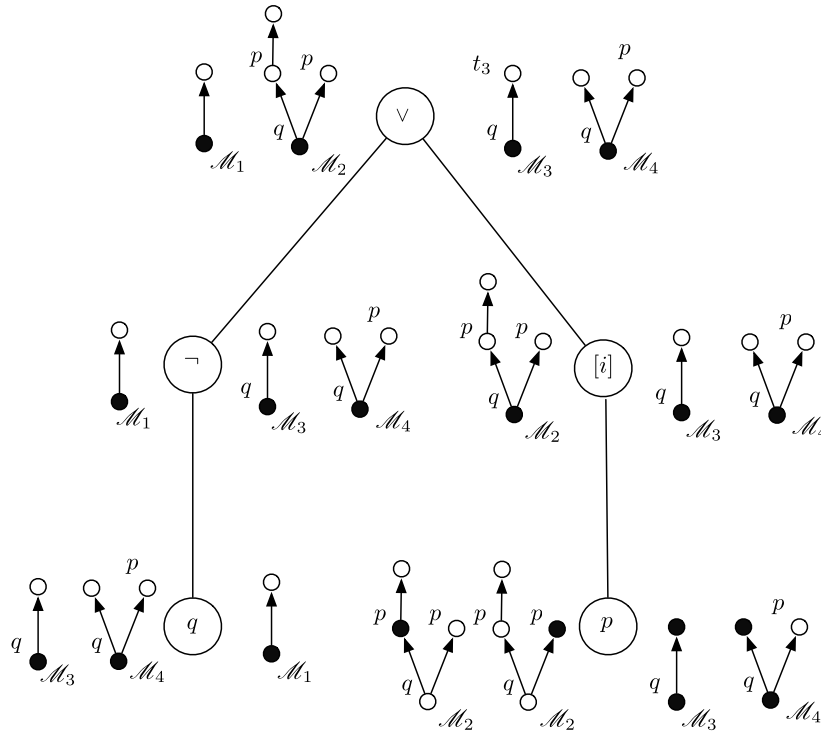


Fig. 1. A 4-round FSG starting in node $\langle \{(\mathcal{M}_1, s_1), (\mathcal{M}_2, s_2)\} \circ \{(\mathcal{M}_3, s_3), (\mathcal{M}_4, s_4)\} \rangle$. A point s in a pointed model (\mathcal{M}, s) is denoted in black, so e.g., s_1 is denoted by the black circle in \mathcal{M}_1 at the top node.

Clearly, suitable game moves can be formulated also for the defined symbols \top , \perp , \wedge , and $\langle i \rangle$. However, this is not essential since we are using FSG to obtain lower bounds on formula size and the introduction of these defined symbols does not lead to a substantial reduction in the size of the formulae that contain them relative to the equivalent formulae built using only the operators \neg , \vee , $[i]$: see Proposition 2.

Example 2. Fig. 1 shows a 4-round FSG starting in a node with two models on each side: $\langle \{(\mathcal{M}_1, s_1), (\mathcal{M}_2, s_2)\} \circ \{(\mathcal{M}_3, s_3), (\mathcal{M}_4, s_4)\} \rangle$. Only the atoms true at a given point are mentioned. There is only one index i . The big circles represent the nodes in the game tree. The current points in the models are solid. Spoiler starts by playing an or-move. He wants to exploit the fact that in one of the models on the left, q is false in the current point, whereas in the other pointed model on the left, all i -successors satisfy p . In the left branch of the or-move, all pointed models on the left satisfy $\neg q$, whereas all on the right satisfy q . In order to close this branch using an atomic-move, Spoiler first needs to swap the models from left to right with a not-move. In the right branch, Spoiler plays an $[i]$ -move: he manages, for every pointed model on the right, to find a successor that fails to verify p , whereas on the left all successors *do* verify p . Note that the latter pointed model gives rise to two pointed models in the successor of the node. In that node, Spoiler can play an atom-move to close this branch. Note that the game tree represents a game that is played along the formula $\neg q \vee [i]p$.

Consider a node $\langle \mathbb{C} \circ \mathbb{D} \rangle$. It is worth noting that the or-move does not specify that Spoiler splits the set \mathbb{C} from the pair into two *disjoint* sets \mathbb{C}_1 and \mathbb{C}_2 . If Spoiler plays according to $\varphi_1 \vee \varphi_2$ for instance, the models in \mathbb{C} that verify both φ_1 and φ_2 may well appear in both \mathbb{C}_1 and \mathbb{C}_2 . Also, it may be that one of those sets \mathbb{C}_1 and \mathbb{C}_2 is empty: the same can happen with \mathbb{C}_1 if it results from \mathbb{C} when Spoiler plays an $[i]$ -move at $\langle \mathbb{C} \circ \mathbb{D} \rangle$: this takes care of the case that no pointed model in \mathbb{C} has an i -successor.

Definition 11 (Isomorphism of branches). A branch B in a closed game tree is any path leading from the root of the tree to a closed leaf.

1. Let two, not necessarily distinct, closed game trees T_1 and T_2 be given and let the branch B_1 , consisting of the nodes $\eta_0, \eta_1, \dots, \eta_k$, and the branch B_2 , consisting of the nodes $\eta'_0, \eta'_1, \eta'_2, \dots, \eta'_l$ (where the nodes in each branch have been numbered in increasing order starting from the root of the tree), belong to T_1 and T_2 , respectively. We call B_1 and B_2 *isomorphic*, and write $B_1 \cong B_2$, iff $k = l$ and the symbols from the set $\Sigma = \{p, \neg, \vee, [i]\}$ labelling the nodes η_j, η'_j ($0 \leq j \leq k$) are the same.
2. For any branch B , let $I(B) = \pi_1 \pi_2 \dots \pi_n$ be a word over I such that π_1 is the index of the first index-move occurring along B (also for index moves, we assume that they have been numbered in increasing order starting from the root of the tree), π_2 is the index of the second index-move, etc. For instance, the branch B (right) of the game tree of Fig. 1

satisfies $I(B) = i$. If we have a game tree induced by the formula $[a](p \vee [b]([a]q \vee \neg[c]\neg p))$, then this tree has three branches where $I(B_1) = a$, $I(B_2) = aba$ and $I(B_3) = abc$.

Intuitively, isomorphism of branches means that the branches are equally long and they look the same provided that we do not take into account the two sets of pointed models labelling the respective nodes η_j, η'_j ($0 \leq j \leq k$). It is obvious that if a game tree T has two non-isomorphic branches B_1 and B_2 , then T has at least two different branches.

Let $Br(T)$ denote the set of branches of a closed game tree T .

Definition 12 (*Isomorphism of game-trees*). Two closed game-trees T_1 and T_2 are called isomorphic, written $T_1 \cong T_2$, if and only if they are both parse trees of the same formula from ML.

The reader may think about this notion in the following way. If we have two closed game-trees that look identical provided that we do not take into account the sets of pointed models labelling each node, then these trees are called isomorphic.

Some properties of formula-size games are listed in the next lemma. Note that only the second item has a “modal” flavour. The rest are general properties that apply to any logic.⁴

We assume the reader is familiar with the notion of bisimulation and the fact that two image-finite pointed models are modally equivalent if and only if they are bisimilar, i.e., the Hennessy–Milner theorem. For the relevant definitions, we refer to [3]. We are going to use the following notation. For any pair of pointed models $\langle \mathbb{A}, \mathbb{B} \rangle$, the set of all closed game trees with root $\langle \mathbb{A} \circ \mathbb{B} \rangle$ is denoted $\mathcal{T}(\langle \mathbb{A} \circ \mathbb{B} \rangle)$.

Lemma 2 (*Properties of FSG*). For any FSG starting at a node $\langle \mathbb{A} \circ \mathbb{B} \rangle$,

1. If there are two bisimilar models $(\mathcal{A}, w) \in \mathbb{A}$ and $(\mathcal{B}, v) \in \mathbb{B}$, then Spoiler cannot win this game in any number of steps.
2. If \mathbb{A} and \mathbb{B} are finite sets of image-finite models and Spoiler cannot win this game in any number of steps, then there are two bisimilar models $(\mathcal{A}, w) \in \mathbb{A}$ and $(\mathcal{B}, v) \in \mathbb{B}$.
3. If $\mathbb{A} = \emptyset$ and $\mathbb{B} \neq \emptyset$, then Spoiler can win this game by playing according to any formula φ such that $\mathbb{B} \models \neg\varphi$.
4. If $\mathbb{A} \neq \emptyset$ and $\mathbb{B} = \emptyset$, then Spoiler can win this game by playing according to any formula φ such that $\mathbb{A} \models \varphi$.
5. If $\mathbb{A} = \mathbb{B} = \emptyset$, then Spoiler can win this game in any number of steps $n \geq 1$.
6. If Spoiler can win this game in n moves, then he can also win in n moves the FSG starting at $\langle \mathbb{A}_1 \circ \mathbb{B}_1 \rangle$, where for every $\mathcal{A}_1 \in \mathbb{A}_1$, there is an $\mathcal{A} \in \mathbb{A}$ that is bisimilar to \mathcal{A}_1 and, similarly, if $\mathcal{B}_1 \in \mathbb{B}_1$, then there is a $\mathcal{B} \in \mathbb{B}$ that is bisimilar to \mathcal{B}_1 ; what is more, for every $T \in \mathcal{T}(\langle \mathbb{A} \circ \mathbb{B} \rangle)$, there is a $T_1 \in \mathcal{T}(\langle \mathbb{A}_1 \circ \mathbb{B}_1 \rangle)$ such that $T \cong T_1$.
7. Suppose that Spoiler can win this game in n moves. Let $\mathbb{A}_1 \subseteq \mathbb{A}$, $\mathbb{B}_1 \subseteq \mathbb{B}$ and let k be the smallest possible number of moves that Spoiler needs to win the FSG starting at $\langle \mathbb{A}_1 \circ \mathbb{B}_1 \rangle$, then $k \leq n$.

Proof. 1. Since bisimilar models satisfy the same formulae of ML, there is no formula φ such that $\mathbb{A} \models \varphi$ and $\mathbb{B} \models \neg\varphi$. Therefore, using Theorem 1, we see that Spoiler cannot win the FSG starting at the node $\langle \mathbb{A} \circ \mathbb{B} \rangle$ in any number of steps $n \geq 1$.

2. Let \mathbb{A} and \mathbb{B} be finite sets of image-finite models and suppose that Spoiler cannot win the game. Then we claim that there are two models $(\mathcal{A}, w) \in \mathbb{A}$ and $(\mathcal{B}, v) \in \mathbb{B}$ such that for all formulae $\varphi \in \Phi_{ML}$,

$$(\mathcal{A}, w) \models \varphi \quad \text{if and only if} \quad (\mathcal{B}, v) \models \varphi$$

Let us assume otherwise. Let $\mathbb{A} = \{(\mathcal{A}_1, w_1) \dots (\mathcal{A}_n, w_n)\}$ and $\mathbb{B} = \{(\mathcal{B}_1, v_1) \dots (\mathcal{B}_k, v_k)\}$. According to our assumption, for every pair of models $(\mathcal{A}_i, w_i) \in \mathbb{A}$ and $(\mathcal{B}_j, v_j) \in \mathbb{B}$, there is a formula κ_{-j}^{+i} such that

$$(\mathcal{A}_i, w_i) \models \kappa_{-j}^{+i} \quad \text{and} \quad (\mathcal{B}_j, v_j) \models \neg\kappa_{-j}^{+i}$$

For $1 \leq i \leq n$, define $\alpha_i = \bigwedge_{1 \leq j \leq k} \kappa_{-j}^{+i}$. By definition of κ_{-j}^{+i} and α_i , we have $(\mathcal{A}_i, w_i) \models \alpha_i$ but $(\mathcal{B}_j, v_j) \models \neg\alpha_i$ (cf. Table 1). It is then obvious that

$$\mathbb{A} \models \bigvee_{1 \leq i \leq n} \alpha_i \quad \text{while} \quad \mathbb{B} \models \neg \bigvee_{1 \leq i \leq n} \alpha_i$$

Using Theorem 1, we see that Spoiler can win the game starting at $\langle \mathbb{A} \circ \mathbb{B} \rangle$ which is a contradiction. Therefore, there are two models $(\mathcal{A}, w) \in \mathbb{A}$ and $(\mathcal{B}, v) \in \mathbb{B}$ that agree on all formulae $\varphi \in \Phi_{ML}$. Since (\mathcal{A}, w) and (\mathcal{B}, v) are image-finite, we can apply the Hennessy–Milner theorem. Therefore, (\mathcal{A}, w) and (\mathcal{B}, v) are bisimilar.

⁴ This applies to items 1 and 6, too. We can replace the word “bisimilar” with any notion that captures the fact that the models satisfy the same formulae e.g., “elementary equivalent”, and 1 and 6 will remain true.

Table 1

$(\mathcal{A}_i, w_i) \models \kappa_{-j}^{+i}$, but $(\mathcal{B}_j, v_j) \models \neg \kappa_{-j}^{+i}$
 $(i \leq n, j \leq k)$.

$(\mathcal{A}_1, w_1) \models \kappa_{-1}^{+1}, (\mathcal{B}_1, v_1) \models \neg \kappa_{-1}^{+1}$
$(\mathcal{A}_1, w_1) \models \kappa_{-2}^{+1}, (\mathcal{B}_2, v_2) \models \neg \kappa_{-2}^{+1}$
\vdots
$(\mathcal{A}_1, w_1) \models \kappa_{-k}^{+1}, (\mathcal{B}_k, v_k) \models \neg \kappa_{-k}^{+1}$
$(\mathcal{A}_2, w_2) \models \kappa_{-1}^{+2}, (\mathcal{B}_1, v_1) \models \neg \kappa_{-1}^{+2}$
$(\mathcal{A}_2, w_2) \models \kappa_{-2}^{+2}, (\mathcal{B}_2, v_2) \models \neg \kappa_{-2}^{+2}$
\vdots
$(\mathcal{A}_2, w_2) \models \kappa_{-k}^{+2}, (\mathcal{B}_k, v_k) \models \neg \kappa_{-k}^{+2}$
\vdots
$(\mathcal{A}_n, w_n) \models \kappa_{-1}^{+n}, (\mathcal{B}_1, v_1) \models \neg \kappa_{-1}^{+n}$
$(\mathcal{A}_n, w_n) \models \kappa_{-2}^{+n}, (\mathcal{B}_2, v_2) \models \neg \kappa_{-2}^{+n}$
\vdots
$(\mathcal{A}_n, w_n) \models \kappa_{-k}^{+n}, (\mathcal{B}_k, v_k) \models \neg \kappa_{-k}^{+n}$

3. Suppose $\mathbb{B} \models \neg \varphi$. Since $\mathbb{A} = \emptyset$, φ is trivially true in all the pointed models in \mathbb{A} . Hence, Spoiler can win the FSG starting at $\langle \mathbb{A} \circ \mathbb{B} \rangle$ by playing according to φ .

4. Suppose $\mathbb{A} \models \varphi$. Since $\mathbb{B} = \emptyset$, φ is trivially false in all the pointed models in \mathbb{B} . Hence, Spoiler can win the FSG starting at $\langle \mathbb{A} \circ \mathbb{B} \rangle$ by playing according to φ .

5. Let $\mathbb{A} = \mathbb{B} = \emptyset$. Then, for any formula ψ , it is trivially true that $\mathbb{A} \models \psi$ and $\mathbb{B} \models \neg \psi$. Therefore, if $|\psi| = n$, then Spoiler can play according to ψ and win the FSG starting at $\langle \mathbb{A} \circ \mathbb{B} \rangle$ in n moves. It is obvious that for any natural number $n \geq 1$ we can find a formula of length n . One such formula is $\neg^{n-1} p$, i.e., $n-1$ occurrences of \neg followed by the propositional symbol p .

6. Since for every $\mathcal{A}_1 \in \mathbb{A}_1$, there is an $\mathcal{A} \in \mathbb{A}$ that is bisimilar to \mathcal{A}_1 and, similarly, for \mathbb{B}_1 and \mathbb{B} , we have that, for all formulae ψ , $\mathbb{A} \models \psi \Rightarrow \mathbb{A}_1 \models \psi$ and $\mathbb{B} \models \psi \Rightarrow \mathbb{B}_1 \models \psi$. Suppose that Spoiler can win the FSG starting at $\langle \mathbb{A} \circ \mathbb{B} \rangle$ in n moves. Then, according to Theorem 1, there is a formula φ of length n such that $\mathbb{A} \models \varphi$ and $\mathbb{B} \models \neg \varphi$. By the observation above, $\mathbb{A}_1 \models \varphi$ whereas $\mathbb{B}_1 \models \neg \varphi$. Therefore, Spoiler can win the FSG starting at $\langle \mathbb{A}_1 \circ \mathbb{B}_1 \rangle$ in n moves by playing according to φ . Moreover, every $T \in \mathcal{T}(\langle \mathbb{A} \circ \mathbb{B} \rangle)$ is a parse tree of some formula ψ such that $\mathbb{A} \models \psi$ and $\mathbb{B} \models \neg \psi$, and by the argument just given, there is a game tree $T_1 \in \mathcal{T}(\langle \mathbb{A}_1 \circ \mathbb{B}_1 \rangle)$ (T_1 being the parse tree of the same ψ) for which $T \cong T_1$.

7. Suppose that $k > n$. It follows immediately from the previous item that Spoiler can win the FSG starting at $\langle \mathbb{A}_1 \circ \mathbb{B}_1 \rangle$ in n moves. Therefore, k is not the minimal number of moves that Spoiler needs to win the FSG starting at $\langle \mathbb{A}_1 \circ \mathbb{B}_1 \rangle$. \square

The formula size games will be the main tool for obtaining our exponential succinctness results formulated in the next sections. There, we apply the general recipe based on Lemma 1 that can be informally described as follows. Suppose that we want to prove that the logic $L_1 = \langle \Phi_1, \models_1, \mathbb{M} \rangle$ is exponentially more succinct than the logic $L_2 = \langle \Phi_2, \models_2, \mathbb{M} \rangle$ on \mathbb{M} .⁵ Then, we can apply the following strategy.

1. All the results in this paper concern cases where $L_2 = \text{ML}$. So we can use our version of Adler–Immerman games for L_2 as defined in Definition 9 and use the result of Theorem 1.
2. For every $n \in \mathbb{N}$, find a formula $\varphi_n \in \Phi_1$ such that $|\varphi_n| = f(n)$, where $f(x)$ is some strictly increasing function.
3. For every $n \in \mathbb{N}$, find two sets of pointed models $\mathbb{A}_n, \mathbb{B}_n \subseteq \mathbb{M}$, such that $\mathbb{A}_n \models_1 \varphi_n$ and $\mathbb{B}_n \models_1 \neg \varphi_n$ and prove that Spoiler cannot win $\mathcal{G}(L_2)$ starting at $\langle \mathbb{A}_n \circ \mathbb{B}_n \rangle$ in less than $2^{g(n)}$ moves, where $g(x)$ grows asymptotically at least as fast as $f(x)$.

Then, all the conditions from Lemma 1 are fulfilled and hence, $L_1 \not\leq_{\mathbb{M}}^{\text{SUBEXP}} L_2$.

The two main difficulties in this strategy come from item 3. The first one is finding the right models which requires an intuitive understanding of the type of properties that are expressed more succinctly by L_1 than by L_2 . For example, it is clear that a formula of the form $\neg[\forall_{\{a,b\}}] \neg p$ is equivalent to the formula $\langle a \rangle p \vee \langle b \rangle p$. Therefore, intuitively, the first formula expresses more efficiently that we can make either an a or b -step from the current point and reach a point that satisfies p . Hence, by stacking n boxes, as in the formula

$$\neg \underbrace{[\forall_{\{a,b\}}] \dots [\forall_{\{a,b\}}]}_{n \text{ times}} \neg p$$

⁵ Of course, we assume that $L_1 \leq_{\mathbb{M}} L_2$.

we actually describe a number of different paths starting at the current point such that at least one of them leads to a point that satisfies p , and moreover, this number is exponential in n . Indeed, this is one of the properties we are going to exploit later. The second difficulty is proving lower bounds on the number of moves in Adler–Immerman games. This is an underdeveloped area and there are currently just two known techniques which we explain below using examples from modal logic. We would like to stress however that the main ideas are applicable to any other logic.

Diverging Pairs: This technique was introduced in [1]. It can be roughly explained as follows. Let us assume that Spoiler can win the game starting at $\langle\{\langle\mathcal{M}_1, w_1\rangle, \langle\mathcal{M}_2, w_2\rangle\} \circ \{\langle\mathcal{N}_1, v_1\rangle, \langle\mathcal{N}_2, v_2\rangle\}\rangle$. We want to know the size (i.e., number of nodes) of the smallest closed game tree T with root $\langle\{\langle\mathcal{M}_1, w_1\rangle, \langle\mathcal{M}_2, w_2\rangle\} \circ \{\langle\mathcal{N}_1, v_1\rangle, \langle\mathcal{N}_2, v_2\rangle\}\rangle$. However, when proving lower bounds, we are not really interested in the precise size of T but whether it is below or above some threshold. One possible answer then is not to look for the precise number of nodes but for the number of different branches. We can try to prove for example that T has at least two branches. One way to do this is the following. If for any pair of trees $T_1 \in \mathcal{T}(\langle\langle\mathcal{M}_1, w_1\rangle \circ \langle\mathcal{N}_1, v_1\rangle\rangle)$ and $T_2 \in \mathcal{T}(\langle\langle\mathcal{M}_2, w_2\rangle \circ \langle\mathcal{N}_2, v_2\rangle\rangle)$, we can prove that there are two branches $B_1 \in \text{Br}(T_1)$ and $B_2 \in \text{Br}(T_2)$, such that $B_1 \not\cong B_2$, then, applying Lemma 2(6), we obtain the desired result (see also Theorem 2). We call the pairs $\langle\langle\mathcal{M}_1, w_1\rangle, \langle\mathcal{N}_1, v_1\rangle\rangle$ and $\langle\langle\mathcal{M}_2, w_2\rangle, \langle\mathcal{N}_2, v_2\rangle\rangle$ diverging because, intuitively, Spoiler cannot “keep them in the same branch” if he wants to win the game starting at $\langle\{\langle\mathcal{M}_1, w_1\rangle, \langle\mathcal{M}_2, w_2\rangle\} \circ \{\langle\mathcal{N}_1, v_1\rangle, \langle\mathcal{N}_2, v_2\rangle\}\rangle$. It is obvious that the number of mutually divergent pairs gives a lower bound on the branches of T and, therefore, its size, too.

Weight Function: The weight-function technique was introduced in [13]. The main idea is the following. Suppose that Spoiler can win the game starting at $\langle\mathbb{A} \circ \mathbb{B}\rangle$. Again, we want to prove a lower bound on the size of the smallest $T \in \mathcal{T}(\langle\mathbb{A} \circ \mathbb{B}\rangle)$. One way to do this is to define a weight function $w : 2^K \times 2^K \rightarrow \mathbb{R}^+$ with the following properties. $w = \langle\langle\mathbb{A}, \mathbb{B}\rangle\rangle = n \geq 0$ and for any pair of sets of pointed models $\langle\mathbb{C}, \mathbb{D}\rangle$,

1. $w(\langle\mathbb{C}, \mathbb{D}\rangle) = w(\langle\mathbb{D}, \mathbb{C}\rangle)$;
2. $w(\langle\mathbb{C}, \mathbb{D}\rangle) = w(\langle\mathbb{C}_1, \mathbb{D}\rangle) + w(\langle\mathbb{C}_2, \mathbb{D}\rangle)$, where $\mathbb{C}_1 \subseteq \mathbb{C}$, and $\mathbb{C}_2 \subseteq \mathbb{C}$;
3. $w(\langle\mathbb{C}_1, \mathbb{D}_1\rangle) = w(\langle\mathbb{C}, \mathbb{D}\rangle) - 1$, where \mathbb{C}_1 and \mathbb{D}_1 are defined as in the last item of Definition 9;
4. If $\mathbb{C} \models p$ and $\mathbb{D} \models \neg p$ for some propositional symbol p , then $w(\langle\mathbb{C}, \mathbb{D}\rangle) = 0$.

Let T be an arbitrary closed game tree with root $\langle\mathbb{A} \circ \mathbb{B}\rangle$. The weight of the root of T is n , the weight of each leaf is 0. Items 1, 2, and 3 specify how the weight of nodes increases from the leaves to the root, when not-, or-, and $[i]$ -moves are played. Then, it is obvious that each closed tree with weight n will have at least n nodes where $[i]$ -moves were played.

We would like to stress that this artificial and very simple example was chosen so that we can showcase the main idea behind the weight function technique. We invite the reader to consult [13] for a technically sophisticated application of this method to proving lower bounds on the size of first-order formulae on the class of linear orders.

All our proofs in the next sections are based on the diverging-pairs technique and that is why we formulate it here explicitly in its full generality.

Theorem 2 (Principle of diverging pairs). *Let $T \in \mathcal{T}(\langle\mathbb{A} \circ \mathbb{B}\rangle)$. If $\mathbb{A}_1, \mathbb{A}_2, \dots, \mathbb{A}_k$ are subsets of \mathbb{A} and $\mathbb{B}_1, \mathbb{B}_2, \dots, \mathbb{B}_k$ are subsets of \mathbb{B} and for every k trees $T_1 \in \mathcal{T}(\langle\mathbb{A}_1 \circ \mathbb{B}_1\rangle), T_2 \in \mathcal{T}(\langle\mathbb{A}_2 \circ \mathbb{B}_2\rangle), \dots, T_k \in \mathcal{T}(\langle\mathbb{A}_k \circ \mathbb{B}_k\rangle)$, there are k branches $B_1 \in \text{Br}(T_1), B_2 \in \text{Br}(T_2), \dots, B_k \in \text{Br}(T_k)$ such that $B_i \not\cong B_j$ for all $1 \leq i < j \leq k$, then T contains at least k different branches.*

Proof. Suppose that $T \in \mathcal{T}(\langle\mathbb{A} \circ \mathbb{B}\rangle)$. It follows from Lemma 2(6) that for $1 \leq i \leq k$, there is a $T_i \in \mathcal{T}(\langle\mathbb{A}_i \circ \mathbb{B}_i\rangle)$ such that $T \cong T_i$. According to the assumption, there are k branches $B_i \in \text{Br}(T_i)$ such that $B_j \not\cong B_l$ for $1 \leq j < l \leq k$. Therefore, there are k branches $B'_i \in \text{Br}(T)$ such that $B'_i \cong B_i$. Hence, T has at least k different branches. \square

3. Main results

Let $\Sigma = \langle A, I \rangle$ be some arbitrary but fixed signature. We consider the logics $\text{ML} = \langle \Phi_{\text{ML}}, \models, \mathbb{M} \rangle$, $[\forall_I]\text{ML} = \langle \Phi_{[\forall_I]\text{ML}}, \models, \mathbb{M} \rangle$, $[\exists_I]\text{ML} = \langle \Phi_{[\exists_I]\text{ML}}, \models, \mathbb{M} \rangle$, and $[\varphi]\text{ML} = \langle \Phi_{[\varphi]\text{ML}}, \models, \mathbb{M} \rangle$ in the signature Σ , where $\mathbb{M} \subseteq \mathbf{K}$. Since I is finite, it is easy to see that for any pointed model $(\mathcal{M}, w) \in \mathbb{M}$,

$$(\mathcal{M}, w) \models [\forall_I]\varphi \quad \text{if and only if} \quad (\mathcal{M}, w) \models \bigwedge_{i \in I} [i]\varphi; \quad (3)$$

$$(\mathcal{M}, w) \models [\exists_I]\varphi \quad \text{if and only if} \quad (\mathcal{M}, w) \models \bigvee_{i \in I} [i]\varphi. \quad (4)$$

Hence, ML , $[\forall_I]\text{ML}$, and $[\exists_I]\text{ML}$ are equally expressive on every $\mathbb{M} \subseteq \mathbf{K}$, i.e., $\text{ML} =_{\mathbb{M}} [\forall_I]\text{ML} =_{\mathbb{M}} [\exists_I]\text{ML}$. Using the well-known equivalences below [21], we see that $\text{ML} =_{\mathbb{M}} [\varphi]\text{ML}$, too: occurrences of $[\varphi]$ in $[\varphi]\psi$ can be ‘pushed inside’ ψ to the propositional atoms, and when such an atom is reached, the occurrence of $[\varphi]$ can be omitted:

Table 2

Formulae in $[\forall_\Gamma]\text{ML}$ and $[\exists_\Gamma]\text{ML}$, with $\Gamma = \{a, b\}$.

	$[\forall_\Gamma]\text{ML}$	$[\exists_\Gamma]\text{ML}$
1.	$\neg[\forall_\Gamma]\neg p$	$[\exists_\Gamma]p$
2.	$\neg[\forall_\Gamma][\forall_\Gamma]\neg p$	$[\exists_\Gamma][\exists_\Gamma]p$
\vdots		
n .	$\neg \underbrace{[\forall_\Gamma] \dots [\forall_\Gamma]}_{n \text{ times}} \neg p$	$\underbrace{[\exists_\Gamma] \dots [\exists_\Gamma]}_{n \text{ times}} p$
\vdots		

- $(\mathcal{M}, w) \models [\varphi]p$ if and only if $(\mathcal{M}, w) \models \varphi \rightarrow p$;
 $(\mathcal{M}, w) \models [\varphi](\psi_1 \vee \psi_2)$ if and only if $(\mathcal{M}, w) \models [\varphi]\psi_1 \vee [\varphi]\psi_2$;
 $(\mathcal{M}, w) \models [\varphi]\neg\psi$ if and only if $(\mathcal{M}, w) \models \varphi \rightarrow \neg[\varphi]\psi$;
 $(\mathcal{M}, w) \models [\varphi][i]\psi$ if and only if $(\mathcal{M}, w) \models \varphi \rightarrow [i][\varphi]\psi$;
 $(\mathcal{M}, w) \models [\varphi_1][\varphi_2]\psi$ if and only if $(\mathcal{M}, w) \models [\varphi_1 \wedge \varphi_2]\psi$.

Therefore, all these logics are equally expressive on any class of models $\mathbb{M} \subseteq \mathbf{K}$; moreover, it is obvious that:

1. $[\forall_\Gamma]\text{ML} \leq_{\mathbb{M}}^{\text{EXP}} \text{ML}$;
2. $[\exists_\Gamma]\text{ML} \leq_{\mathbb{M}}^{\text{EXP}} \text{ML}$;
3. $[\varphi]\text{ML} \leq_{\mathbb{M}}^{\text{EXP}} \text{ML}$.

Our main result is that the above succinctness statements cannot be improved for the case where \mathbb{M} is replaced by **S5** (the class of models whose relations are relations of equivalence), i.e., any of the logics $[\forall_\Gamma]\text{ML}$, $[\exists_\Gamma]\text{ML}$, $[\varphi]\text{ML}$ is exponentially more succinct than ML on **S5**. Using Proposition 1, we see that we have exponential succinctness results on any class of models $\mathbf{L} \subseteq \mathbf{K}$, such that **S5** $\subseteq \mathbf{L}$. However, we begin by presenting a proof of exponential succinctness on a class of models that is considerably simpler than **S5** first, because it provides us with the opportunity to make the main idea behind all of our proofs explicit in this more transparent setting.

3.1. Succinctness on \mathbf{K}

In this subsection, we work with a signature $\Sigma = \langle A, I \rangle$ such that I contains at least two indices a and b and A contains at least one propositional symbol p . The main result we are going to prove is the following theorem.

Theorem 3. For any logic $L \in \{[\forall_\Gamma]\text{ML}, [\exists_\Gamma]\text{ML}, [\varphi]\text{ML}\}$ in the signature Σ , it is true that $L \not\leq_{\mathbf{K}}^{\text{SUBEXP}} \text{ML}$.

The case $[\varphi]\text{ML} \not\leq_{\mathbf{K}}^{\text{SUBEXP}} \text{ML}$ was proven in [18]. Our proof of the two remaining statements follows the strategy described in the previous section.

Consider the two sequences of formulae, where $\Gamma = \{a, b\}$.

We will show that every formula $\psi_n \in \Phi_{\text{ML}}$ that is equivalent to one of the formulae $\neg \underbrace{[\forall_\Gamma] \dots [\forall_\Gamma]}_{n \text{ times}} \neg p$ or $\underbrace{[\exists_\Gamma] \dots [\exists_\Gamma]}_{n \text{ times}} p$

has length at least 2^n . Since the length of the latter formulae is linear in n , we will have our exponential succinctness result. To show that their equivalents in ML have exponential size, we begin by defining for each $n \geq 1$, two sets of pointed models \mathbb{A}^n and \mathbb{B}^n such that $\mathbb{A}^n \models \neg \underbrace{[\forall_\Gamma] \dots [\forall_\Gamma]}_{n \text{ times}} \neg p$ and $\mathbb{A}^n \models \underbrace{[\exists_\Gamma] \dots [\exists_\Gamma]}_{n \text{ times}} p$ whereas $\mathbb{B}^n \models \underbrace{[\forall_\Gamma] \dots [\forall_\Gamma]}_{n \text{ times}} \neg p$ and $\mathbb{B}^n \models$

$\neg \underbrace{[\exists_\Gamma] \dots [\exists_\Gamma]}_{n \text{ times}} p$.

Definition 13 (Words over an alphabet Δ). Let Δ be a set of symbols. The set of words $\mathcal{W}(\Delta)$ over Δ , and their length, are defined as follows. There is a word of length 0, denoted by $\varepsilon \in \mathcal{W}(\Delta)$. Moreover, if $\delta \in \Delta$ and $w \in \mathcal{W}(\Delta)$ is a word of length n , then $w\delta$ is a word in $\mathcal{W}(\Delta)$ of length $n + 1$. Given a word w of length $k > 0$, with $w = w_1, w_2, \dots, w_k$, we say that $w_1 \in \Delta$ is the first element of the word, and w_k is the last, while w_i is the element at the i -th position in w . By $\mathcal{W}^{\leq n}(\Delta)$ we denote all words in $\mathcal{W}(\Delta)$ of length at most n .

Definition 14 (Tree-models). For any $n \geq 1$ and any word w of length n from $\mathcal{W}(\{a, b\})$, the model $\mathcal{A}_w^n = \langle A^n, R, V_w \rangle$ is constructed as follows:

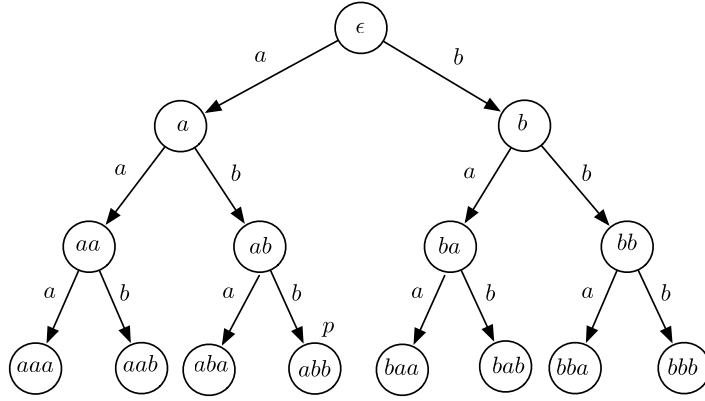


Fig. 2. The model $(\mathcal{A}^3_{abb}, \varepsilon)$.

- $A^n = \mathcal{W}^{\leq n}(\{a, b\})$, the set of words over $\{a, b\}$ of length at most n ;
- uR_iv if and only if $v = ui$;
- $V_w(p) = \{w\}$ and $V_w(q) = \emptyset$ for all other propositional symbols in A .

The set \mathbb{A}^n consists of all the 2^n different pointed models $(\mathcal{A}_w^n, \varepsilon)$. The set \mathbb{B}^n contains the only pointed model $(\mathcal{B}^n, \varepsilon)$, where $\mathcal{B}^n = \langle A^n, R, V_\emptyset \rangle$ is constructed as above with the only exception that $V_\emptyset(p) = \emptyset$.

Fig. 2 shows the model $(\mathcal{A}^3_{abb}, \varepsilon) \in \mathbb{A}^3$. It is obvious that \mathbb{A}^3 contains 8 models: $(\mathcal{A}^3_{aaa}, \varepsilon)$, where $V_{aaa}(p) = aaa$; $(\mathcal{A}^3_{aab}, \varepsilon)$, where $V_{aab}(p) = aab$, etc. The only model in \mathbb{B}^n has no point where p is true.

Proposition 3. For every $n \geq 1$, the following are true.

- (a) $\mathbb{A}^n \models \neg[\forall_I] \dots [\forall_I] \neg p$ and $\mathbb{B}^n \models [\forall_I] \dots [\forall_I] \neg p$;
- (b) $\mathbb{A}^n \models [\exists_I] \dots [\exists_I] p$ and $\mathbb{B}^n \models \neg[\exists_I] \dots [\exists_I] p$.

Proof. The proof of both items is by an easy induction on n . We prove only item (a). If $n = 1$, then \mathbb{A}^1 contains just the two pointed models $(\mathcal{A}_a^1, \varepsilon)$ and $(\mathcal{A}_b^1, \varepsilon)$, whereas \mathbb{B}^1 consists of the single model $(\mathcal{B}^1, \varepsilon)$. Since $(\mathcal{A}_a^1, \varepsilon) \models \langle a \rangle p$ and $(\mathcal{A}_b^1, \varepsilon) \models \langle b \rangle p$ and the formula $\neg[\forall_I] \neg p$ is equivalent to the formula $\langle a \rangle p \vee \langle b \rangle p$, we see that $\mathbb{A}^1 \models \neg[\forall_I] \neg p$. It is obvious that $\mathbb{B}^1 \models [\forall_I] \neg p$. Let us assume that the statement is true for n and let us consider the case $n + 1$. For every model $(\mathcal{A}_w^{n+1}, \varepsilon) \in \mathbb{A}^{n+1}$, we have that $\varepsilon R_a a$ and $\varepsilon R_b b$ and exactly one of the following is true:

- (\mathcal{A}_w^{n+1}, a) is bisimilar to the model $(\mathcal{A}_u^n, \varepsilon) \in \mathbb{A}^n$ where $w = au$ or
- (\mathcal{A}_w^{n+1}, b) is bisimilar to the model $(\mathcal{A}_u^n, \varepsilon) \in \mathbb{A}^n$ where $w = bu$.

At the same time, for the only model $(\mathcal{B}^{n+1}, \varepsilon) \in \mathbb{B}^{n+1}$, we have that $\varepsilon R_a a$ and $\varepsilon R_b b$ and both (\mathcal{B}^{n+1}, a) and (\mathcal{B}^{n+1}, b) are bisimilar to $(\mathcal{B}^n, \varepsilon)$. Since $\neg[\forall_I] \dots [\forall_I] \neg p$ is equivalent to $\langle a \rangle \neg[\forall_I] \dots [\forall_I] \neg p \vee \langle b \rangle \neg[\forall_I] \dots [\forall_I] \neg p$, the statement follows from the induction hypothesis and the items above. \square

Proposition 3 and the fact that $[\forall_I]\text{ML}$, $[\exists_I]\text{ML}$ and ML are equally expressive on every class of models imply that Spoiler can win the formula size game starting at $\langle \mathbb{A}^n \circ \mathbb{B}^n \rangle$. Therefore, in order to complete the proof of Theorem 3, we have to show that the size of the smallest tree $T \in \mathcal{T}(\langle \mathbb{A} \circ \mathbb{B} \rangle)$ is at least 2^n . The next lemma is crucial because it provides us with enough diverging pairs for that.

Lemma 3. For every $(\mathcal{A}_w^n, \varepsilon) \in \mathbb{A}^n$ and every $n \geq 1$, it is true that every tree $T \in \mathcal{T}((\mathcal{A}_w^n, \varepsilon) \circ (\mathcal{B}^n, \varepsilon))$ has a branch B such that $I(B) = w$.

Proof. The fact that Spoiler can win the game starting at $\langle \mathbb{A}^n \circ \mathbb{B}^n \rangle$ and Lemma 2(6) imply that $\mathcal{T}((\mathcal{A}_w^n, \varepsilon) \circ (\mathcal{B}^n, \varepsilon))$ is non-empty, which in turn implies that $\mathcal{T}((\mathcal{B}^n, \varepsilon) \circ (\mathcal{A}_w^n, \varepsilon))$ is non-empty, too. We prove the stronger statement that any $T \in \mathcal{T}((\mathcal{A}_w^n, \varepsilon) \circ (\mathcal{B}^n, \varepsilon)) \cup \mathcal{T}((\mathcal{B}^n, \varepsilon) \circ (\mathcal{A}_w^n, \varepsilon))$ contains a branch B , such that $I(B) = w$. The proof is by induction on n .

Base case

Let $n = 1$ and $T \in \mathcal{T}((\mathcal{A}_w^1, \varepsilon) \circ (\mathcal{B}^1, \varepsilon)) \cup \mathcal{T}((\mathcal{B}^1, \varepsilon) \circ (\mathcal{A}_w^1, \varepsilon))$, where w is either a or b . The required branch B is defined inductively. The first point of B is the root of T . It is obvious that Spoiler cannot begin the game with an atomic move. Therefore, he starts by playing either a not-move, an or-move or an $[i]$ -move. If a not-move is played, then either the node $((\mathcal{B}^1, \varepsilon) \circ (\mathcal{A}_w^1, \varepsilon))$ or the node $((\mathcal{A}_w^1, \varepsilon) \circ (\mathcal{B}^1, \varepsilon))$ is added to the game tree. Let this be the second node of the branch B . If an or-move was originally played, at least one node $((\mathcal{A}_w^1, \varepsilon) \circ (\mathcal{B}^1, \varepsilon))$ or $((\mathcal{B}^1, \varepsilon) \circ (\mathcal{A}_w^1, \varepsilon))$ was added. Let this be the second node of B . It is obvious that Spoiler still cannot play an atomic move at a node $((\mathcal{B}^1, \varepsilon) \circ (\mathcal{A}_w^1, \varepsilon))$ or $((\mathcal{A}_w^1, \varepsilon) \circ (\mathcal{B}^1, \varepsilon))$. Hence, Spoiler again plays either a not-move, an or-move or an $[i]$ -move. If Spoiler this time again plays a not-move or an or-move, we repeat the procedure above and add the third node to B which is either $((\mathcal{B}^1, \varepsilon) \circ (\mathcal{A}_w^1, \varepsilon))$ or $((\mathcal{A}_w^1, \varepsilon) \circ (\mathcal{B}^1, \varepsilon))$. It is obvious that this branch cannot end in a closed leaf if Spoiler does not play at least one $[i]$ -move such that $i \in \{a, b\}$.

Playing an $[i]$ -move at a node $((\mathcal{B}^1, \varepsilon) \circ (\mathcal{A}_w^1, \varepsilon))$ or $((\mathcal{A}_w^1, \varepsilon) \circ (\mathcal{B}^1, \varepsilon))$ where $i \neq w$ leads to loss for Spoiler because both nodes $((\mathcal{B}^1, i) \circ (\mathcal{A}_w^1, i))$ and $((\mathcal{A}_w^1, i) \circ (\mathcal{B}^1, i))$ contain bisimilar models, one on the left and one on the right. Hence, Spoiler plays an $[w]$ -move at $((\mathcal{B}^1, \varepsilon) \circ (\mathcal{A}_w^1, \varepsilon))$ or $((\mathcal{A}_w^1, \varepsilon) \circ (\mathcal{B}^1, \varepsilon))$ thus adding the node $((\mathcal{B}^1, w) \circ (\mathcal{A}_w^1, w))$ or $((\mathcal{A}_w^1, w) \circ (\mathcal{B}^1, w))$ to the tree, respectively. We add this node to B . Note that no $[i]$ -moves can be played at the nodes $((\mathcal{B}^1, w) \circ (\mathcal{A}_w^1, w))$ and $((\mathcal{A}_w^1, w) \circ (\mathcal{B}^1, w))$. Therefore, all paths leading from this node to a closed leaf contain just not-moves, or-moves or atomic-moves. Hence we have the desired branch B and the statement is true for $n = 1$.

Induction step

Assume now that the statement is true for n . Let $T \in \mathcal{T}((\mathcal{A}_w^{n+1}, \varepsilon) \circ (\mathcal{B}^{n+1}, \varepsilon)) \cup \mathcal{T}((\mathcal{B}^{n+1}, \varepsilon) \circ (\mathcal{A}_w^{n+1}, \varepsilon))$, where $w = jk$ with k a word over $\{a, b\}$ of length n and $j \in \{a, b\}$. The desired branch B is again defined inductively with its first node being the root of T . The reasoning we used in the base case shows that the root of T lies on a branch in which the first $[i]$ -move is a $[j]$ -move played at a node $((\mathcal{A}_{jk}^{n+1}, \varepsilon) \circ (\mathcal{B}^{n+1}, \varepsilon))$ or $((\mathcal{B}^{n+1}, \varepsilon) \circ (\mathcal{A}_{jk}^{n+1}, \varepsilon))$ thus leading to the node $((\mathcal{A}_{jk}^{n+1}, j) \circ (\mathcal{B}^{n+1}, j))$ or $((\mathcal{B}^{n+1}, j) \circ (\mathcal{A}_{jk}^{n+1}, j))$, respectively. Since T is closed, its subtree T_1 with root $((\mathcal{A}_{jk}^{n+1}, j) \circ (\mathcal{B}^{n+1}, j))$ or $((\mathcal{B}^{n+1}, j) \circ (\mathcal{A}_{jk}^{n+1}, j))$ must also be closed. We add the root of T_1 to B . Given the fact that $(\mathcal{A}_{jk}^{n+1}, j)$ is bisimilar to $(\mathcal{A}_k^n, \varepsilon)$ and (\mathcal{B}^{n+1}, j) is bisimilar to $(\mathcal{B}^n, \varepsilon)$ we can apply Lemma 2(6). Therefore, there is a $T' \in \mathcal{T}((\mathcal{A}_k^n, \varepsilon) \circ (\mathcal{B}^n, \varepsilon)) \cup \mathcal{T}((\mathcal{B}^n, \varepsilon) \circ (\mathcal{A}_k^n, \varepsilon))$, such that $T_1 \cong T'$ and we can apply the induction hypothesis. \square

Now we are ready to complete the proof of Theorem 3. Using Lemma 3, we see that for every $n \geq 1$, the pair $\langle \mathbb{A}^n, \mathbb{B}^n \rangle$ contains 2^n diverging pairs. It follows from Theorem 2 that every $T \in \mathcal{T}(\langle \mathbb{A}^n, \mathbb{B}^n \rangle)$ is of size at least 2^n . Therefore, the proof is complete.

As an aside, note that our proof in fact demonstrates something stronger than Theorem 3: we have shown that both $[\forall_I]\text{ML}$ and $[\exists_I]\text{ML}$ are exponentially more succinct than ML, even on the class of binary trees $\mathbf{BT} \subseteq \mathbf{K}$ (where the underlying relation on the binary trees is $R_a \cup R_b$).

3.2. Succinctness on $\mathbf{S5}$

For $\mathbf{M} \subseteq \mathbf{K}$, results like Theorem 3 are more difficult to prove, since there are more candidates for formulae to be equivalent to either formulae in $[\forall_I]\text{ML}$ or $[\exists_I]\text{ML}$. Consider once again the models $(A_w^3, \varepsilon) = \langle A^3, R, V_s \rangle$ from \mathbf{K} as depicted in Fig. 2 for instance. Suppose we are interested in proving a similar result on models where $R(i)$ are equivalence relations and let therefore $Cl(R(i))$ be the smallest equivalence relation that includes $R(i)$. We would then have that the models \mathbb{A}^3 on the left of the game tree would all satisfy $\langle a \rangle \langle b \rangle \langle a \rangle \langle b \rangle \langle a \rangle \langle b \rangle p$, which is linear in n : the new relations allow transitions labelled a or b that stay in the same point (or even go ‘back’).

We proceed now to our main result, namely, the following theorem.

Theorem 4. Let $\Sigma = \langle A, I \rangle$ be a signature such that I contains at least 4 indices $\{a, b, c, d\}$ and A contains at least 3 propositional symbols $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$. For any logic $L \in \{[\forall_I]\text{ML}, [\exists_I]\text{ML}, [\varphi]\text{ML}\}$ in the signature Σ , it is true that $L \not\leq_{\mathbf{S5}}^{\text{SUBEXP}} \text{ML}$.

The proof of this theorem follows the same strategy as before. Let $\Gamma = \{a, b\}$ and let φ_0 be the formula $\langle a \rangle \mathbf{a} \vee \langle b \rangle \mathbf{b}$.⁶ Consider the sequences of formulae $\vartheta_n, \sigma_n, \varphi_n$ ($n = 1, 2, \dots$) as given in Table 3. As an aside, note that we use conjunction and diamonds $\langle i \rangle$ ($i \in I$) and $\langle \varphi \rangle$ in some of the definitions, but we know from Proposition 2 that this does not affect the succinctness results stated in Theorem 4.

It is easy to see that the lengths of ϑ_n, σ_n and φ_n are linear in n . We will prove that every ML-formula that is equivalent to one of these formulae has length at least 2^n .

⁶ In this section, it is convenient to use the same symbols for propositional variables and indices. Still, to distinguish them in writing, we use boldface for variables from A .

Table 3
Formulae.

$[\forall_I]\text{ML}$		$[\exists_I]\text{ML}$		$[\varphi]\text{ML}$	
ϑ_1	$\neg[\forall_I]\neg\mathbf{c}$	σ_1	$[\exists_I]\mathbf{c}$	φ_1	$\langle \mathbf{c} \rangle (\mathbf{c} \wedge \langle d \rangle (\mathbf{d} \wedge \varphi_0))$
\vdots		\vdots		\vdots	
ϑ_n	$\neg[\forall_I][c][d]\neg\vartheta_{n-1}$	σ_n	$[\exists_I](\mathbf{c} \wedge [c][d]\sigma_{n-1})$	φ_n	$\langle \varphi_{n-1} \rangle \varphi_1$
\vdots		\vdots		\vdots	

3.2.1. Defining suitable models

We are now going to define sets of pointed models, on which we will play FSGs. More precisely, we build sets \mathbb{A}^n and \mathbb{B}^n to show that the equivalent in ML of ϑ_n is exponential in size, sets \mathbb{C}^n and \mathbb{D}^n for the equivalents in ML of σ_n , and \mathbb{E}^n and \mathbb{F}^n for those equivalent to φ_n . To talk about all those three cases in general terms, let the pair $(\mathbb{X}^n, \mathbb{Y}^n)$ be a variable over $\{(\mathbb{A}^n, \mathbb{B}^n), (\mathbb{C}^n, \mathbb{D}^n), (\mathbb{E}^n, \mathbb{F}^n)\}$.

From now on, all models will be **S5**-models, i.e., the underlying relations are equivalence relations. In our drawings, we will not include reflexive edges, and, since relations are now symmetric, we will not use arrows when denoting edges. When conceptualising our models to be defined next, it may be useful for the reader to think of them in terms of ladders. The following terminology will also be used in their description. We introduce this terminology on the basis of the model on the left of Fig. 4. We will refer to the point \hat{x}_2 as the left foot of the ladder, and point \hat{x}_2 as the right foot. Similarly, \hat{y}_1 is the left top, and \hat{y}_1 the right top. The points $\hat{x}_2, \hat{y}_2, \hat{z}_2, \hat{x}_1, \hat{y}_1$ form the left support of the ladder, the points $\hat{x}_2, \hat{y}_2, \hat{z}_2, \hat{x}_1, \hat{y}_1$ form the right support. The left support is comprised of a path $acda$ of indices. This is a path from \hat{x}_2 to \hat{y}_1 : there are of course many other paths between those points (recall that there is a reflexive edge between any point and itself). Horizontal edges in the ladder will be called *rungs*. For instance, the points \hat{z}_2 and \hat{z}_2 form the left and right end of a rung that is labelled with the indices $I \setminus \{c, d\}$.

For all our models, points at the left support of the ladder are taken from $\{\hat{x}_i, \hat{y}_i, \hat{z}_i\}$, while points at the right support are from $\{\hat{x}_i, \hat{y}_i, \hat{z}_i\}$. A point \hat{x}_i is only horizontally connected, and hence forms a rung, with \hat{x}_i . Likewise for \hat{y}_i and \hat{y}_i , and for \hat{z}_i and \hat{z}_i . Hence, if we use \hat{p} as an arbitrary point on the left support, by \hat{p} we mean the only point that forms a rung with it, on the right support. The classes of models \mathbb{C} and \mathbb{D} are ladders with a ‘third, middle support’: the points on this support are indicated as \bar{x}_i, \bar{y}_i and \bar{z}_i (see Fig. 6). On such models, we will still say that \hat{x}_i, \hat{x}_i and \bar{x}_i are horizontally connected, so are \hat{y}_i, \hat{y}_i and \bar{y}_i , and another horizontal rung is formed by \hat{z}_i, \hat{z}_i and \bar{z}_i . On such models, if \hat{p} is a point on the left support, then \hat{p} is connected to it through a rung on the right support, and \bar{p} is the point connected to it on the middle of the rung.

Recall that $\{a, b, c, d\} \subseteq I$. The pairs of sets of models $(\mathbb{X}^n, \mathbb{Y}^n) \in \{(\mathbb{A}^n, \mathbb{B}^n), (\mathbb{C}^n, \mathbb{D}^n), (\mathbb{E}^n, \mathbb{F}^n)\}$ are all such that for a given n , we define 2^n different models \mathcal{M}_w^n , one for each word w , and we add one pointed model $(\mathcal{M}_w^n, \hat{x}_n)$ to \mathbb{X}^n , and another pointed model $(\mathcal{M}_w^n, \hat{x}_n)$ in \mathbb{Y}^n . The word w is such that it encodes a path from \hat{x}_n , the left foot of the ladder, to the left top. This path uses n steps labelled with indices from $\Gamma = \{a, b\}$. The indices $\{c, d\}$ are used to ensure that every two occurrences of an index from Γ are separated by at least one occurrence of cd in that path. So we will not use all words to generate models: for the models in $\mathbb{A}^n, \mathbb{B}^n, \mathbb{C}^n$ and \mathbb{D}^n , we use words from $\mathcal{W}_{\Gamma(cd\Gamma)^*}^n$ (see Definition 15), while for the models in \mathbb{E}_2^n and \mathbb{F}_w^n we use words from $\mathcal{W}_{(cd\Gamma)^*}^n$ (Definition 19). In any model, for any two points \hat{p} and \hat{p} on the left and right support respectively, there is a step ‘up’ labelled i from \hat{p} iff there is a step ‘up’ labelled i from \hat{p} . Moreover, the only i ’s that qualify for such a vertical transition are indices from $\{a, b, c, d\}$, and, for every point \hat{p} and \hat{p} , it is exactly one index that labels a transition ‘up’. Points \hat{p} and \hat{p} on a rung are connected using all indices j such that there is no vertical transition possible from \hat{p} or \hat{p} with the label j . (We’ll explain later when, in the models $\mathbb{C}^n \cup \mathbb{D}^n$, the three points on a rung \hat{p}, \bar{p} and \hat{p} are connected.)

In terms of the formula size games, they will be played on the node $\langle \mathbb{X}^n \circ \mathbb{Y}^n \rangle$. We now know \mathbb{X}^n will contain 2^n models, and for each model $(\mathcal{M}, \hat{x}_n) \in \mathbb{X}^n$ there is a model $(\mathcal{M}, \hat{x}_n) \in \mathbb{Y}^n$. Consider a node in the game with (\mathcal{M}, \hat{x}_n) on one side, and (\mathcal{M}, \hat{x}_n) at the other. The models are constructed in such a way, that for any \hat{p} on the left support, if a game node contains a model (\mathcal{M}, \hat{p}) on one side of the node, and (\mathcal{M}, \hat{p}) at the other, Spoiler cannot play an $[i]$ -move if \hat{p} and \hat{p} are connected with a rung labelled i in the model: doing so would yield a node in the game tree with two bisimilar models on one side. At the same time, except for the top points in the ladder \mathcal{M} , all points \hat{p} and \hat{p} verify the same propositional variables. The left top in the models in $\mathbb{A}^n \cup \mathbb{B}^n \cup \mathbb{C}^n \cup \mathbb{D}^n$ is the point \hat{y}_1 : it satisfies a designated atom \mathbf{c} , whereas the right top point is \hat{y}_1 : this does not satisfy \mathbf{c} . In the models $\mathbb{E}^n \cup \mathbb{F}^n$, the left top point is \hat{z}_0 , while the top right point is \hat{z}_0 . The points \hat{z}_0 satisfy either \mathbf{a} or \mathbf{b} , while the points \hat{z}_0 satisfy neither of those. All this together implies for the formula size game with a node with (\mathcal{M}, \hat{x}_n) on one side, and (\mathcal{M}, \hat{x}_n) at the other, that if Spoiler wins this game then there is a branch such that, for every \hat{p} in the left support of \mathcal{M} , the models (\mathcal{M}, \hat{p}) and (\mathcal{M}, \hat{p}) will appear each at a side of a node in the branch, and the branch can be only closed once (\mathcal{M}, \hat{q}) and (\mathcal{M}, \hat{q}) are reached, where \hat{q} and \hat{q} are the left and right top of the ladder defined by \mathcal{M} , respectively.

Definition 15 ($\Gamma(cd\Gamma)^*$ -alternating words). For each $n \geq 1$, the set $\mathcal{W}_{\Gamma(cd\Gamma)^*}^n$ of $\Gamma(cd\Gamma)^*$ -alternating words over the alphabet $\{a, b, c, d\}$ is defined as follows:

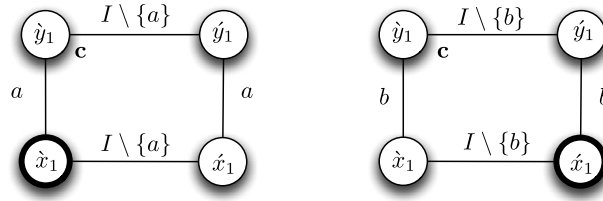


Fig. 3. The models $(\mathcal{A}_a^1, \dot{x}_1)$ (left) and $(\mathcal{A}_b^1, \dot{x}_1)$ (right).

- $\mathcal{W}_{\Gamma(cd\Gamma)^*}^1 = \{a, b\} = \Gamma$;
- $\mathcal{W}_{\Gamma(cd\Gamma)^*}^{n+1} = \{acd w \mid w \in \mathcal{W}_{\Gamma(cd\Gamma)^*}^n\} \cup \{bcd w \mid w \in \mathcal{W}_{\Gamma(cd\Gamma)^*}^n\}$.

So $\Gamma(cd\Gamma)^*$ -alternating words are words over $\{a, b, c, d\}$ of length $3n - 2$, with a c occurring at every position $2 + 3k$, a d at every position $3k$, and at every other position there is one element from $\{a, b\}$. A word $w \in \mathcal{W}_{\Gamma(cd\Gamma)^*}^n$ has n occurrences of a 's and b 's, and every two occurrences are separated by a number of cd 's, cf. $w = acdbcdcbcd \in \mathcal{W}_{\Gamma(cd\Gamma)^*}^4$. The notation $\mathcal{W}_{\Gamma(cd\Gamma)^*}^{n+1}$ is inspired by the fact that the set contains expressions of the form $\Gamma; (cd\Gamma)^n$, that is, an element from Γ , followed by n iterations of the form $cd\gamma$, with $\gamma \in \Gamma$.

Definition 16 (Closure). If W is a set and $R \subseteq W \times W$ a relation, we denote the reflexive, transitive and symmetric closure of R by $Cl(R)$.

We now define the sets of models \mathcal{A}^n and \mathcal{B}^n . The reader may wish to combine reading the definition together with the text that explains it, which follows the definition.

Definition 17 (\mathbb{A}^n and \mathbb{B}^n). The two sets of pointed model $\mathbb{A}^n, \mathbb{B}^n$ are constructed by induction on n .

- $\mathbb{A}^1 = \{(\mathcal{A}_w^1, \dot{x}_1) \mid w \in \mathcal{W}_{\Gamma(cd\Gamma)^*}^1\}$ and $\mathbb{B}^1 = \{(\mathcal{A}_w^1, \dot{x}_1) \mid w \in \mathcal{W}_{\Gamma(cd\Gamma)^*}^1\}$ where $\mathcal{A}_w^1 = \langle A^1, R_w^1, V^1 \rangle$ is such that (see also Fig. 3):
 - $A^1 = \{\dot{x}_1, \dot{x}_1, \dot{y}_1, \dot{y}_1\}$;
 - For any $i \in I$, if $i = w$, then $R_w^1(i)$ is $Cl(\{(\dot{x}_1, \dot{y}_1), (\dot{x}_1, \dot{y}_1)\})$; if $i \neq w$, then $R_w^1(i)$ is $Cl(\{(\dot{y}_1, \dot{y}_1), (\dot{x}_1, \dot{x}_1)\})$;
 - $V(c) = \{\dot{y}_1\}$ and $V(p) = \emptyset$ for any $p \in A$ such that $p \neq c$.
- Let $n > 1$. $\mathbb{A}^n = \{(\mathcal{A}_w^n, \dot{x}_n) \mid w \in \mathcal{W}_{\Gamma(cd\Gamma)^*}^n\}$ and $\mathbb{B}^n = \{(\mathcal{A}_w^n, \dot{x}_n) \mid w \in \mathcal{W}_{\Gamma(cd\Gamma)^*}^n\}$ where $\mathcal{A}_w^n = \langle A^n, R_w^n, V^n \rangle$ is such that (see also Fig. 4):
 - $A^n = A^{n-1} \cup \{\dot{x}_n, \dot{x}_n, \dot{y}_n, \dot{y}_n, \dot{z}_n, \dot{z}_n\}$;
 - R_w^n is defined as follows. Since $w \in \mathcal{W}_{\Gamma(cd\Gamma)^*}^n$, it is of the form $jcdw'$, where $j \in \{a, b\}$, and $w' \in \mathcal{W}_{\Gamma(cd\Gamma)^*}^{n-1}$.
 - * Then $R_w^n(j) = Cl(R_{w'}^{n-1}(j) \cup R_j)$, with $R_j = \{(\dot{x}_n, \dot{y}_n), (\dot{x}_n, \dot{y}_n), (\dot{z}_n, \dot{z}_n)\}$;
 - * $R_w^n(c) = Cl(R_{w'}^{n-1}(c) \cup R_c)$, with $R_c = \{(\dot{y}_n, \dot{z}_n), (\dot{y}_n, \dot{z}_n), (\dot{x}_n, \dot{x}_n)\}$;
 - * $R_w^n(d) = Cl(R_d \cup (R_{w'}^{n-1}(d) \setminus \{(\dot{x}_{n-1}, \dot{x}_{n-1}), (\dot{x}_{n-1}, \dot{x}_{n-1})\}))$, with $R_d = \{(\dot{z}_n, \dot{x}_{n-1}), (\dot{z}_n, \dot{x}_{n-1}), (\dot{y}_n, \dot{y}_n), (\dot{x}_n, \dot{x}_n)\}$;
 - * For any $i \notin \{j, c, d\}$, $R_w^n(i) = Cl(R_{w'}^{n-1}(i) \cup R_i)$, with $R_i = \{(\dot{x}_n, \dot{x}_n), (\dot{y}_n, \dot{y}_n), (\dot{z}_n, \dot{z}_n)\}$.
 - $V^n = V^{n-1}$.

Intuitively, in the pointed model $(\mathcal{A}_w^n, \dot{x}_n)$, the word w encodes the shortest path from \dot{x}_n to \dot{y}_1 , the only point in the model that makes c true. The two models of Fig. 3 show one model from \mathbb{A}^1 (left) and one from \mathbb{B}^1 (right): the points with a thick edge denote the point of the pointed model. Note that both \mathbb{A}^1 and \mathbb{B}^1 contain each two models.

Fig. 3 shows a pointed model $(\mathcal{A}_a^1, \dot{x}_1) \in \mathbb{A}^1$ and $(\mathcal{A}_b^1, \dot{x}_1) \in \mathbb{B}^1$, where Fig. 4 shows pointed models $(\mathcal{A}_w^2, \dot{x}_2) \in \mathbb{A}^2$ and $(\mathcal{A}_{w'}^2, \dot{x}_2) \in \mathbb{B}^2$ with $w = acda \in \mathcal{W}_{cd}^2$. Given n and $w = w_1 w_2 \dots w_n$, the left support of the ladder comprises of the points $\dot{x}_n, \dot{y}_n, \dot{z}_n, \dot{x}_{n-1}, \dots, \dot{y}_2, \dot{z}_2, \dot{x}_1, \dot{y}_1$ (note there is no \dot{z}_1) and w encodes a path along those points: one can go from \dot{x}_n , the foot of the left rail, to \dot{y}_1 , the top of that rail, by the edges labelled w_1, w_2, \dots, w_n . There is exactly the same path to travel from \dot{x}_n , the foot of the right support, to the top of that support, \dot{y}_1 . In every \dot{x}_j and \dot{x}_j ($1 \leq j \leq n$) the only way up is through an index from Γ , in every \dot{y}_j and \dot{y}_j ($1 \leq j \leq n$), the only way up is through c , and in every \dot{z}_j and \dot{z}_j ($1 \leq j \leq n$), the only way up is through d . Two models \mathcal{A}_w^n and $\mathcal{A}_{w'}^n$ with $w \neq w'$ differ in the following sense: there must be at least a $j \geq 2$ such that in one model, the points \dot{x}_j and \dot{y}_{j-1} are connected by the index a (in that model, \dot{x}_j and \dot{y}_{j-1} are connected by a as well), while in the other model, these points are connected by the index b .

The rungs of the ladder connect either pairs \dot{x}_i and \dot{x}_i , or \dot{y}_i and \dot{y}_i ($1 \leq i \leq n$) or pairs \dot{z}_j and \dot{z}_j ($2 \leq j \leq n$). A rung between two points s and t is labelled with all indices that are not used as a label on any other edge with s as a vertex. In any model, \dot{y}_1 is the only point where c is true. This implies, for any n and $w \in \mathcal{W}_{\Gamma(cd\Gamma)^*}^n$, that the shortest path from any $(\mathcal{A}_w^n, \dot{x}_n)$ to a world satisfying c is of length $3n - 2$, whereas the shortest path from $(\mathcal{A}_w^n, \dot{x}_n)$ to a c -world is of length

$3n - 1$. As a consequence, for every w and n we have $(\mathcal{A}_w^n, \dot{x}_n) \models \vartheta_n$ and $(\mathcal{A}_w^n, \dot{x}_n) \models \neg\vartheta_n$. In Fig. 4 for example, we have $(\mathcal{A}_{acda}^2, \dot{x}_2) \models \neg[\forall_I][c][d][\forall_I]\neg\mathbf{c}$, because $(\mathcal{A}_{acda}^2, \dot{x}_2) \models \langle a \rangle \langle c \rangle \langle d \rangle \langle a \rangle \mathbf{c}$.

The next observation states that, by taking the closure of each accessibility relation R_i , we never add edges that are not part of the ladder construction.

Observation 1. For every n , all models in $\mathbb{A}^n \cup \mathbb{B}^n$ are **S5**-models with the property that, for any two points p and p' in the model, and for any $i \in I$: if $p \neq p'$ and $pR_i p'$, then either (p, p') or (p', p) was explicitly added to R_i in the definition of the model (i.e., they are not added because of the transitive closure).

Proof. Since two pointed models $(\mathcal{A}_w^n, \dot{x}_n)$ in \mathbb{A}^n and $(\mathcal{A}_w^n, \dot{x}_n)$ in \mathbb{B}^n only differ in their specific points \dot{x}_n and \dot{x}_n , the underlying structure in any two models \mathcal{A}_w^n and \mathcal{B}_w^n is the same. So it suffices to prove the observation for \mathbb{A}^n only. This, in turn, can be easily seen using induction on n . The claim is such that it suffices to show that the construction of the model does not force us to add any edge due to transitivity. For $n = 1$, all models in \mathbb{A}^1 are depicted in Fig. 3, and it is easily checked that for each $i \in I$, the relation R_i is an equivalence relation: no edges need to be added. So suppose that \mathbb{A}^n satisfies our observation. Take a model $(\mathcal{A}_w^{n+1}, \dot{x}_{n+1})$, with $w = jcdw'$, for some $j \in \Gamma$ and $w' \in \mathcal{W}_{\Gamma(cd\Gamma)^*}^n$. We need to show that by adding points to $(\mathcal{A}_w^n, \dot{x}_n)$ to obtain $(\mathcal{A}_w^{n+1}, \dot{x}_{n+1})$, we do not have to add edges to restore transitivity. The model \mathcal{A}_w^{n+1} has at the bottom the two points \dot{x}_n and \dot{x}_n . From each of them, one can go ‘up’ using a j step, then a c -step, and then a d -step, after we have reached either \dot{x}_n or \dot{x}_n from which the first step ‘up’ is through some $j' \in \Gamma$. What is important here is that there are no two consecutive steps ‘up’ using the same index, so that no edge needs to be added to restore transitivity. By the induction hypothesis, no edges are added ‘above’ \dot{x}_{n-1} and \dot{x}_{n-1} . Moreover, for any $\dot{p} \in \{\dot{x}_{n+1}, \dot{y}_{n+1}, \dot{z}_{n+1}, \dot{x}_n\}$, the points \dot{p} and \dot{p} are horizontally connected by exactly those indices which cannot be used to go ‘up’ or ‘down’ from \dot{p} or \dot{p} , so that also here there is no situation where we are forced to add an additional edge in order to keep transitivity. \square

Here is another feature of our construction, which we will informally demonstrate using the models of Fig. 4 and which will be formalised in Lemma 5. We will argue that for instance the pairs $((\mathcal{A}_{acda}^2, \dot{x}_2), (\mathcal{A}_{acda}^2, \dot{x}_2))$ and $((\mathcal{A}_{bcda}^2, \dot{x}_2), (\mathcal{A}_{bcda}^2, \dot{x}_2))$ are diverging pairs. To see this, suppose $(\mathcal{A}_{acda}^2, \dot{x}_2), (\mathcal{A}_{bcda}^2, \dot{x}_2)$ occur on one side of a node in the tree of an FSG game, and $(\mathcal{A}_{acda}^2, \dot{x}_2)$ and $(\mathcal{A}_{bcda}^2, \dot{x}_2)$ (the latter model is not shown in Fig. 4) occur on the other side. For Spoiler to win the game, he needs to reach the points \dot{y}_1 , and then play an atom move. We claim that, in order to reach this goal, Spoiler needs to play at some point in the game a move that either splits the models $(\mathcal{A}_{acda}^2, \dot{x}_2)$ and $(\mathcal{A}_{bcda}^2, \dot{x}_2)$ so that they appear at different nodes in the successor nodes of the game, or he has to split the models $(\mathcal{A}_{acda}^2, \dot{x}_2)$ and $(\mathcal{A}_{bcda}^2, \dot{x}_2)$. Thus, Spoiler will need to play an or-move. If he likes, Spoiler can first play some not-moves and swap the models from side, but at some point he will play an $[i]$ -move for some i . There are a number of possibilities. Suppose $(\mathcal{A}_{acda}^2, \dot{x}_2)$ occurs on the right side of the tree, together with $(\mathcal{A}_{bcda}^2, \dot{x}_2)$. If Spoiler were to play a $[b]$ -move in the current node, he would either choose $(\mathcal{A}_{acda}^2, \dot{x}_2)$ as the b -successor itself, or otherwise $(\mathcal{A}_{acda}^2, \dot{x}_2)$. But in both cases, in the resulting node we would obtain a model at the left of the node that is bisimilar to the model thus obtained, namely $(\mathcal{A}_{acda}^2, \dot{x}_2)$ or $(\mathcal{A}_{acda}^2, \dot{x}_2)$, respectively. And we know that if two bisimilar models appear on either side of a node, Spoiler loses. Hence, in the current node, given the presence of $(\mathcal{A}_{acda}^2, \dot{x}_2)$, Spoiler will not play a $[b]$ -move, and, by symmetry, given the presence of $(\mathcal{A}_{bcda}^2, \dot{x}_2)$, he will not play an $[a]$ -move either. This implies he plays an $[i]$ -move for some $i \in I \setminus \{a, b\}$, but it is easy to see that for every i -successor of $(\mathcal{A}_{acda}^2, \dot{x}_2)$ that Spoiler will chose, there is a bisimilar i -successor of $(\mathcal{A}_{bcda}^2, \dot{x}_2)$.

A similar argument demonstrates that the pairs $((\mathcal{A}_{acda}^2, \dot{x}_2), (\mathcal{A}_{acda}^2, \dot{x}_2))$ and $((\mathcal{A}_{acdb}^2, \dot{x}_2), (\mathcal{A}_{acdb}^2, \dot{x}_2))$ are diverging pairs: Spoiler can postpone splitting the pairs by first playing some or-moves that do not split the pairs, not-moves interleaved with an $[a]$ -move, a $[c]$ -move and a $[d]$ -move (this would yield a node in the game tree with the pairs $(\mathcal{A}_{acda}^2, \dot{x}_2), (\mathcal{A}_{acda}^2, \dot{x}_2)$ on one side of the node, and $(\mathcal{A}_{acdb}^2, \dot{x}_2), (\mathcal{A}_{acdb}^2, \dot{x}_2)$ on the other side), but from this node on, before playing a index-move, Spoiler had to split the pairs. This argument will be formalised and proven in Lemmas 5 and 6.

Definition 18 (\mathbb{C}^n and \mathbb{D}^n). The two sets of pointed models $\mathbb{C}^n, \mathbb{D}^n$ are constructed as follows.

- $\mathbb{C}^1 = \{(\mathcal{C}_w^1, \dot{x}_1) \mid w \in \mathcal{W}_{\Gamma(cd\Gamma)^*}^1\}$ and $\mathbb{D}^1 = \{(\mathcal{C}_w^1, \dot{x}_1) \mid w \in \mathcal{W}_{\Gamma(cd\Gamma)^*}^1\}$ where $\mathcal{C}_w^1 = \{C^1, R_w^1, V^1\}$ is such that (see also Fig. 5):
 - $C^1 = \{\bar{x}_1, \dot{x}_1, \dot{x}_1, \dot{y}_1, \dot{y}_1\}$.
 - For any $i \in I$, if $i = w$, then $R_w^1(i) = Cl(\{\langle \dot{x}_1, \dot{y}_1 \rangle, \langle \dot{x}_1, \dot{y}_1 \rangle\})$; if $i \in \mathcal{W}_{\Gamma(cd\Gamma)^*}^1 \setminus \{w\}$, then $R_w^1(i) = Cl(\{\langle \dot{y}_1, \dot{y}_1 \rangle, \langle \dot{x}_1, \dot{x}_1 \rangle, \langle \dot{x}_1, \bar{x}_1 \rangle\})$; and if $i \notin \mathcal{W}_{\Gamma(cd\Gamma)^*}^1$, then $R_w^1(i) = Cl(\{\langle \dot{y}_1, \dot{y}_1 \rangle, \langle \dot{x}_1, \dot{x}_1 \rangle\})$.
 - $V(\mathbf{c}) = C^1 \setminus \{\dot{y}_1, \bar{x}_1\}$ and $V(\mathbf{p}) = \emptyset$ for any $\mathbf{p} \in P$ such that $\mathbf{p} \neq \mathbf{c}$.
- Now let $n > 1$. $\mathbb{C}^n = \{(\mathcal{C}_w^n, \dot{x}_n) \mid w \in \mathcal{W}_{\Gamma(cd\Gamma)^*}^n\}$ and $\mathbb{D}^n = \{(\mathcal{C}_w^n, \dot{x}_1) \mid w \in \mathcal{W}_{\Gamma(cd\Gamma)^*}^n\}$ where $\mathcal{C}_w^n = \{C^n, R_w^n, V^n\}$ is such that (see also Fig. 6):
 - $C^n = C^{n-1} \cup \{\dot{x}_n, \dot{x}_n, \dot{y}_n, \dot{y}_n, \dot{z}_n, \dot{z}_n, \bar{x}_n, \bar{y}_n, \bar{z}_n\}$.
 - R_w^n is defined in the following way. Since $w \in \mathcal{W}_{\Gamma(cd\Gamma)^*}^n$, it is of the form $jcdw'$, where $j \in \{a, b\}$, and $w' \in \mathcal{W}_{\Gamma(cd\Gamma)^*}^{n-1}$.

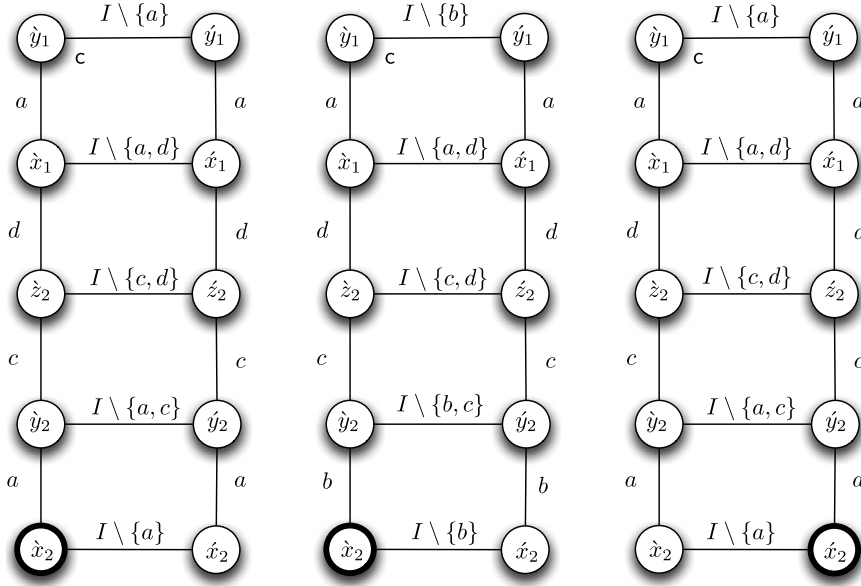


Fig. 4. The models $(\mathcal{A}_{acda}^2, \dot{x}_2)$ (left) and $(\mathcal{A}_{bcda}^2, \dot{x}_2)$ (right) for $w = acda \in \mathcal{W}_{\Gamma(cd\Gamma)^*}^2$. In the middle: $(\mathcal{A}_{acda}^2, \dot{x}_2)$.

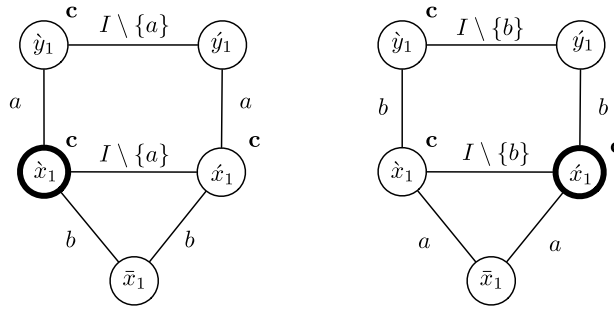


Fig. 5. The models $(\mathcal{C}_a^1, \dot{x}_1)$ (left) and $(\mathcal{C}_b^1, \dot{x}_1)$ (right).

- * $R_w^n(j) = Cl(R_w^{n-1}(j) \cup R_j)$, where $R_j = \{(\dot{x}_n, \dot{y}_n), (\dot{x}_n, \dot{y}_n), (\dot{z}_n, \dot{z}_n)\}$;
- * If $i \in \{a, b\} \setminus \{j\}$, then $R_w^n(i) = Cl(R_w^{n-1}(i) \cup R_i)$, where $R_i = \{(\dot{x}_n, \dot{x}_n), (\dot{x}_n, \dot{x}_n), (\dot{y}_n, \dot{y}_n), (\dot{y}_n, \dot{y}_n), (\dot{z}_n, \dot{z}_n), (\dot{z}_n, \dot{z}_n)\}$;
- * $R_w^n(c) = Cl(R_w^{n-1}(c) \cup R_c)$, where $R_c = \{(\dot{y}_n, \dot{z}_n), (\dot{y}_n, \dot{z}_n), (\dot{x}_n, \dot{x}_n)\}$;
- * $R_w^n(d) = Cl(R_d \cup (R_w^{n-1}(d) \setminus \{(\dot{x}_{n-1}, \dot{x}_{n-1}), (\dot{x}_{-1}n, \dot{x}_{n-1})\}))$, where $R_d = \{(\dot{z}_n, \dot{x}_{n-1}), (\dot{z}_n, \dot{x}_{n-1}), (\dot{y}_n, \dot{y}_n), (\dot{x}_n, \dot{x}_n)\}$;
- * For any $i \in I \setminus \{a, b, c, d\}$, $R_w^n(i) = Cl(R_w^{n-1}(i) \cup R_i)$, where $R_i = \{(\dot{x}_n, \dot{x}_n), (\dot{y}_n, \dot{y}_n), (\dot{z}_n, \dot{z}_n)\}$.
- $V^n(\mathbf{c}) = V^{n-1}(\mathbf{c}) \cup \{\dot{x}_n, \dot{x}_n, \dot{y}_n, \dot{y}_n, \dot{z}_n, \dot{z}_n\}$; $V^n(\mathbf{p}) = \emptyset$ for any $\mathbf{p} \in P$ such that $\mathbf{p} \neq \mathbf{c}$.

Again, in the pointed model $(\mathcal{C}_w^n, \dot{x}_n)$, the word w encodes the shortest path from \dot{x}_n to \dot{y}_1 .

The models $(\mathcal{C}_w^n, \dot{x}_n)$ and $(\mathcal{C}_w^n, \dot{x}_n)$ can be informally described as follows (see also Figs. 5 and 6). They are based on the ladder models $(\mathcal{A}_w^n, \dot{x}_n)$ and $(\mathcal{B}_w^n, \dot{x}_n)$ respectively, but there are two differences. This time, for every rung (except for the top one), we connect the points \dot{p} and \dot{p} on the rung with a third point, \bar{p} . That is, the models $(\mathcal{C}_w^1, \dot{x}_n)$ and $(\mathcal{C}_w^1, \dot{x}_n)$ only have one such point, \bar{x}_1 , and it is connected with \dot{x}_1 and \dot{x}_1 using the index $i \in \{a, b\}$ for which one cannot go ‘up’ from \dot{x}_1 , i.e., with $i = w$. Let $(\dot{p}_n, \dot{p}_n) \in \{(\dot{x}_n, \dot{x}_n), (\dot{y}_n, \dot{y}_n), (\dot{z}_n, \dot{z}_n)\}$ be a pair on a new rung of the ladder model $(\mathcal{C}_w^n, \dot{x}_n)$ or $(\mathcal{C}_w^n, \dot{x}_n)$. Then we add a point $m \in \{\bar{x}_n, \bar{y}_n, \bar{z}_n\}$ and the points \dot{p}_n and \dot{p}_n are connected with \bar{p}_n through an edge labelled with $i \in \{a, b\}$ such that $w \neq jw'$, or in other words, the index from $\{a, b\}$ that is not used to go ‘up’ in \dot{p}_n . The other difference between the models in $\mathbb{C}^n \cup \mathbb{D}^n$ and the models in $\mathbb{A}^n \cup \mathbb{B}^n$ is the valuation for the propositional variable \mathbf{c} : for the models in $(\mathcal{C}_w^n, \dot{x}_n)$ and $(\mathcal{C}_w^n, \dot{x}_n)$ this variable is false in \dot{y}_1 (the right top of the ladder) and in all points in the middle: $\{\bar{x}_1\} \cup \{\bar{x}_i, \bar{y}_i, \bar{z}_i \mid 2 \leq i \leq n\}$.

The following observation can be proven along the lines of Observation 1.

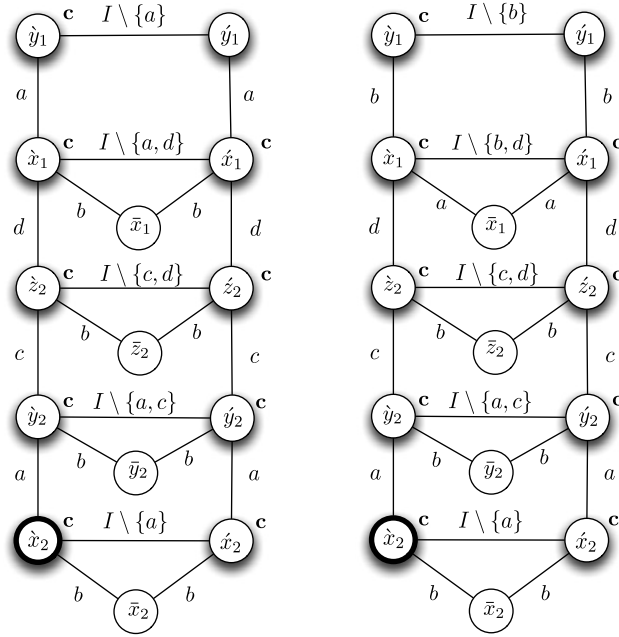


Fig. 6. The models $(\mathcal{C}_{acda}^2, \dot{x}_2)$ (left) and $(\mathcal{C}_{acdb}^2, \dot{x}_2)$ (right).

Observation 2. For every n , all models in $\mathbb{C}^n \cup \mathbb{D}^n$ are **S5**-models with the property that, for any two points p and p' in the model, and for any $i \in I$: if $p \neq p'$ and $pR_i p'$, then either (p, p') or (p', p) was explicitly added to R_i in the definition of the model (i.e., they are not added because of the transitive closure).

Proposition 4. Recall that $\Gamma = \{a, b\}$. For all $n \geq 1$, it is true that

1. $\mathbb{A}^n \models \vartheta_n$ and $\mathbb{B}^n \models \neg\vartheta_n$;
2. $\mathbb{C}^n \models \sigma_n$ and $\mathbb{D}^n \models \neg\sigma_n$.

Proof. We prove the slightly more complicated second item. The proof of item 1 is analogous. The proof works as follows. First of all, we will quickly verify that all points in the middle of the ladder (the points $\bar{x}_i, \bar{y}_i, \bar{z}_i$) all verify $\neg\sigma_j$, for all $1 \leq j \leq n$. Then, for all σ_j ($1 \leq j \leq n$) and all points p_i at a support of the ladder, i.e., for $p_i \in \{\dot{x}_i, \dot{x}_i, \dot{y}_i, \dot{y}_i, \dot{z}_i, \dot{z}_i\}$, we show that all σ_j with $j < i$ are true in p_i , while \dot{x}_j verifies σ_j , but \dot{x}_j does not.

Let us consider an arbitrary pointed model $\mathcal{C}_w^n = \{\mathcal{C}^n, R_w^n, V^n\}$. Note that $\neg\sigma_{n+1}$, which is $\neg[\exists_{\{a,b\}}](\mathbf{c} \wedge [c][d]\sigma_n)$ is equivalent to $\langle a \rangle(\neg\mathbf{c} \vee \langle c \rangle \langle d \rangle \neg\sigma_n) \wedge \langle b \rangle(\neg\mathbf{c} \vee \langle c \rangle \langle d \rangle \neg\sigma_n)$. Since all relations in our models are reflexive and for every middle point $m \in \{\bar{x}_1\} \cup \{\bar{x}_i, \bar{y}_i, \bar{z}_i \mid 2 \leq i \leq n\}$ it is true that $(\mathcal{C}_w^n, m) \models \neg\mathbf{c}$, it follows immediately that for all $j \geq 1$, we have $(\mathcal{C}_w^n, m) \models \neg\sigma_j$.

Next we show that

- A:** for every $1 \leq j \leq n$, it is true that
- A1:** if $j < i \leq n$, and $p_i \in \{\dot{x}_i, \dot{x}_i, \dot{y}_i, \dot{y}_i, \dot{z}_i, \dot{z}_i\}$ then $(\mathcal{C}_w^n, p_i) \models \sigma_j$, and
- A2:** if $j = i$, then $(\mathcal{C}_w^n, \dot{x}_i) \models \sigma_j$, whereas $(\mathcal{C}_w^n, \dot{x}_i) \models \neg\sigma_j$.

The proof is by induction on j .

Base case

Let $j = 1$. It is obvious that σ_1 is equivalent to $[a]\mathbf{c} \vee [b]\mathbf{c}$. Suppose that $1 < i$. It follows from the definition of the models \mathcal{C}_w^n that $\{\dot{x}_i, \dot{x}_i, \dot{y}_i, \dot{y}_i, \dot{z}_i, \dot{z}_i\} \subseteq V^n(\mathbf{c})$; moreover, there is an $l \in \{a, b\}$ such that $\dot{x}_i R_l \dot{y}_i$, $\dot{x}_i R_l \dot{y}_i$ and there is no point p such that $(\mathcal{C}_w^n, p) \models \neg\mathbf{c}$ and p is an l -successor of one of the points $\{\dot{x}_i, \dot{x}_i, \dot{y}_i, \dot{y}_i\}$. Hence,

- $(\mathcal{C}_w^n, \dot{x}_i) \models \sigma_1$;
- $(\mathcal{C}_w^n, \dot{x}_i) \models \sigma_1$;
- $(\mathcal{C}_w^n, \dot{y}_i) \models \sigma_1$;
- $(\mathcal{C}_w^n, \dot{y}_i) \models \sigma_1$.

Let us consider now the points \dot{z}_i and \dot{z}_i . Again, the construction of the model \mathcal{C}_w^n is such that there is an $l \in \{a, b\}$, such that $\dot{z}_i R_l \dot{z}_i$ and there is no point p that is an l -successor of one of these points and at the same time $(\mathcal{C}_w^n, p) \models \neg\mathbf{c}$.

Therefore, $(\mathcal{C}_w^n, \hat{z}_i) \models \sigma_1$ and $(\mathcal{C}_w^n, \hat{z}_i) \models \sigma_1$. To complete the base case, we have to prove **A2** which says that $(\mathcal{C}_w^n, \hat{x}_1) \models \sigma_1$ and $(\mathcal{C}_w^n, \hat{x}_1) \models \neg\sigma_1$. The construction of the models is such that $\{\hat{y}_1, \hat{x}_1\} \subseteq V^n(\mathbf{c})$ and $\{\hat{x}_1, \hat{y}_1\} \cap V^n(\mathbf{c}) = \emptyset$; moreover, $\hat{x}_1 R_l \hat{y}_1$ and $\hat{x}_1 R_k \hat{x}_1$ for some $l \neq k$ and $l, k \in \{a, b\}$. Hence, $(\mathcal{C}_w^n, \hat{x}_1) \models \langle a \rangle \neg \mathbf{c} \wedge \langle b \rangle \neg \mathbf{c}$, i.e., $(\mathcal{C}_w^n, \hat{x}_1) \models \neg\sigma_1$. At the same time $\hat{x}_1 R_l \hat{y}_1$, and there is no point p such that $\hat{x}_1 R_l p$ and $p \notin V^n(\mathbf{c})$. Therefore, $(\mathcal{C}_w^n, \hat{x}_1) \models [a]\mathbf{c} \vee [b]\mathbf{c}$, i.e., $(\mathcal{C}_w^n, \hat{x}_1) \models \sigma_1$.

Induction step

Suppose that $j + 1 < i \leq n$ and let us assume that the statement **A** is true for j . It is obvious that σ_{j+1} is equivalent to the formula $[a](\mathbf{c} \wedge [c][d]\sigma_j) \vee [b](\mathbf{c} \wedge [c][d]\sigma_j)$. We first prove that **A1** is true for the points \hat{x}_n and \hat{y}_n . All the other cases are analogous.

According to the induction hypothesis, it is true that

- $(\mathcal{C}_w^n, \hat{x}_n) \models \sigma_j$ and $(\mathcal{C}_w^n, \hat{x}_n) \models \sigma_j$;
- $(\mathcal{C}_w^n, \hat{y}_n) \models \sigma_j$ and $(\mathcal{C}_w^n, \hat{y}_n) \models \sigma_j$;
- $(\mathcal{C}_w^n, \hat{z}_n) \models \sigma_j$ and $(\mathcal{C}_w^n, \hat{z}_n) \models \sigma_j$;
- $(\mathcal{C}_w^n, \hat{x}_{n-1}) \models \sigma_j$ and $(\mathcal{C}_w^n, \hat{x}_{n-1}) \models \sigma_j$.

Since $\hat{x}_n R_d \hat{x}_n$, $\hat{x}_n R_c \hat{x}_n$ and there is no other point p that is either a d - or a c -successor of \hat{x}_n or \hat{x}_n and at the same time $(\mathcal{C}_w^n, p) \models \neg\sigma_j$, we see that $(\mathcal{C}_w^n, \hat{x}_n) \models \mathbf{c} \wedge [c][d]\sigma_j$ and $(\mathcal{C}_w^n, \hat{x}_n) \models \mathbf{c} \wedge [c][d]\sigma_j$. On the other hand, the induction hypothesis and the fact that $\hat{y}_n R_d \hat{y}_n$, and there is no other point p that is a d -successor of either \hat{y}_n or \hat{y}_n and $(\mathcal{C}_w^n, p) \models \neg\sigma_j$ imply that $(\mathcal{C}_w^n, \hat{y}_n) \models [d]\sigma_j$ and $(\mathcal{C}_w^n, \hat{y}_n) \models [d]\sigma_j$. In a similar fashion, the induction hypothesis and the fact that $\hat{z}_n R_d \hat{x}_{n-1}$, $\hat{z}_n R_d \hat{x}_{n-1}$, and there is no point p that is a d -successor of either \hat{z}_n or \hat{z}_n and $(\mathcal{C}_w^n, p) \models \neg\sigma_j$ imply that $(\mathcal{C}_w^n, \hat{z}_n) \models [d]\sigma_j$ and $(\mathcal{C}_w^n, \hat{z}_n) \models [d]\sigma_j$. Given the shape of the model \mathcal{C}_w^n , we obtain that $(\mathcal{C}_w^n, \hat{y}_n) \models \mathbf{c} \wedge [c][d]\sigma_j$ and $(\mathcal{C}_w^n, \hat{y}_n) \models \mathbf{c} \wedge [c][d]\sigma_j$. We know that for some $l \in \{a, b\}$, \hat{y}_n is an l -successor of \hat{x}_n , \hat{y}_n is an l -successor of \hat{x}_n and there is no other point p that is an l -successor of either \hat{x}_n or \hat{x}_n and $(\mathcal{C}_w^n, p) \models \neg(\mathbf{c} \wedge [c][d]\sigma_j)$. Therefore, $(\mathcal{C}_w^n, \hat{x}_n) \models [l](\mathbf{c} \wedge [c][d]\sigma_j)$ and $(\mathcal{C}_w^n, \hat{x}_n) \models [l](\mathbf{c} \wedge [c][d]\sigma_j)$. Hence, $(\mathcal{C}_w^n, \hat{x}_n) \models \sigma_{j+1}$ and $(\mathcal{C}_w^n, \hat{x}_n) \models \sigma_{j+1}$.

The fact that $(\mathcal{C}_w^n, \hat{x}_n) \models \sigma_n$ can be established in the same way as above. Next, we show **A2**, i.e., that $(\mathcal{C}_w^n, \hat{x}_n) \models \neg\sigma_n$. According to the induction hypothesis, $(\mathcal{C}_w^n, \hat{x}_{n-1}) \models \neg\sigma_{n-1}$. Additionally, we have $\hat{x}_n R_l \hat{y}_n$ for some $l \in \{a, b\}$, $\hat{y}_n R_c \hat{z}_n$, and $\hat{z}_n R_d \hat{x}_{n-1}$. Therefore, $(\mathcal{C}_w^n, \hat{x}_{n-1}) \models [l](\neg \mathbf{c} \vee \langle c \rangle \langle d \rangle \neg\sigma_{n-1})$.

On the other hand, we know already that $(\mathcal{C}_w^n, \hat{x}_n) \models \neg\sigma_{n-1}$. Given the shape of the model \mathcal{C}_w^n , there is a $k \in \{a, b\}$ such that $k \neq l$ and $\hat{x}_n R_k \hat{x}_n$. Hence, $(\mathcal{C}_w^n, \hat{x}_{n-1}) \models [k](\neg \mathbf{c} \vee \langle c \rangle \langle d \rangle \neg\sigma_{n-1})$ and, therefore, $(\mathcal{C}_w^n, \hat{x}_n) \models \neg\sigma_n$. \square

We now turn our attention to the formulae φ_n , and the models $\mathbb{E}^n \cup \mathbb{F}^n$ for them. Before defining those models though, we show that, indeed, the sequence of formulae φ_n has equivalents ψ_n in ML. This was easily seen for the sequences using ϑ_n and σ_n , and, although this equivalence is not needed for the sequel (after all, we are interested in the *shortest* formulae χ_n in ML equivalent to φ_n), it may give the reader an additional way of looking at the models to be defined.

Proposition 5. Define $\psi_n \in \text{ML}$ as follows:

$$\begin{aligned} \psi_1 &= \langle c \rangle (\mathbf{c} \wedge \langle d \rangle (\mathbf{d} \wedge (\langle a \rangle \mathbf{a} \vee \langle b \rangle \mathbf{b}))) \\ \psi_{n+1} &= \psi_n \wedge \langle c \rangle (\psi_n \wedge \mathbf{c} \wedge \langle d \rangle (\psi_n \wedge \mathbf{d} \wedge (\langle a \rangle (\psi_n \wedge \mathbf{a}) \vee \langle b \rangle (\psi_n \wedge \mathbf{b})))) \end{aligned}$$

Then, even on \mathbf{K} , for all n , φ_n and ψ_n are equivalent.

Proof. First note that, by definition of φ_1 and φ_0 , we have

$$\varphi_1 = \langle c \rangle (\mathbf{c} \wedge \langle d \rangle (\mathbf{d} \wedge (\langle a \rangle \mathbf{a} \vee \langle b \rangle \mathbf{b}))) \quad (5)$$

Which settles the claim for $n = 1$. Now suppose, we have proven that φ_n and ψ_n are equivalent on \mathbf{K} .

Our next step is to show that for all $n \geq 1$ we have:

$$\langle \varphi_n \rangle \varphi_1 \equiv (\varphi_n \wedge \langle c \rangle (\varphi_n \wedge \mathbf{c} \wedge \langle d \rangle (\varphi_n \wedge \mathbf{d} \wedge (\langle a \rangle (\varphi_n \wedge \mathbf{a}) \vee \langle b \rangle (\varphi_n \wedge \mathbf{b})))))) \quad (6)$$

We use the following equivalences for public announcements, for more details of those, see [25, Chapter 4]. Let φ and ψ be arbitrary formulae, $e \in I$ and $\mathbf{e} \in A$. Then

$$\langle \varphi \rangle \psi \equiv \varphi \wedge \langle \varphi \rangle \psi \quad (7)$$

$$\langle \varphi \rangle \langle e \rangle \psi \equiv \varphi \wedge \langle e \rangle \langle \varphi \rangle \psi \quad (8)$$

$$\langle \varphi \rangle (\psi_1 \wedge \psi_2) \equiv \langle \varphi \rangle \psi_1 \wedge \langle \varphi \rangle \psi_2 \quad (9)$$

$$\langle \varphi \rangle (\psi_1 \vee \psi_2) \equiv \langle \varphi \rangle \psi_1 \vee \langle \varphi \rangle \psi_2 \quad (10)$$

$$\langle \varphi \rangle \mathbf{e} \equiv \varphi \wedge \mathbf{e} \quad (11)$$

Using (7), (8) and (11), we get

$$\langle \varphi \rangle \langle e \rangle (\mathbf{e} \wedge \psi) \equiv \varphi \wedge \langle e \rangle (\varphi \wedge \mathbf{e} \wedge \langle \varphi \rangle \psi) \quad (12)$$

We then obtain the following equivalences, using (5) and (12), respectively:

$$\langle \varphi_n \rangle \varphi_1 \equiv \langle \varphi_n \rangle \langle c \rangle (\mathbf{c} \wedge \langle d \rangle (\mathbf{d} \wedge (\langle a \rangle \mathbf{a} \vee \langle b \rangle \mathbf{b}))) \quad (13)$$

$$\equiv \varphi_n \wedge \langle c \rangle (\varphi_n \wedge \mathbf{c} \wedge \langle \varphi_n \rangle \langle d \rangle (\mathbf{d} \wedge (\langle a \rangle \mathbf{a} \vee \langle b \rangle \mathbf{b}))) \quad (14)$$

We also have, by (12) again,

$$\langle \varphi_n \rangle \langle d \rangle (\mathbf{d} \wedge (\langle a \rangle \mathbf{a} \vee \langle b \rangle \mathbf{b})) \equiv \varphi_n \wedge \langle d \rangle (\varphi_n \wedge \mathbf{d} \wedge \langle \varphi_n \rangle (\langle a \rangle \mathbf{a} \vee \langle b \rangle \mathbf{b})) \quad (15)$$

Using (10) and (12) (with $\psi = \top$), we also have

$$\langle \varphi_n \rangle (\langle a \rangle \mathbf{a} \vee \langle b \rangle \mathbf{b}) \equiv \langle \varphi_n \rangle \langle a \rangle \mathbf{a} \vee \langle \varphi_n \rangle \langle b \rangle \mathbf{b} \quad (16)$$

$$\equiv (\varphi_n \wedge \langle a \rangle (\varphi_n \wedge \mathbf{a})) \vee (\varphi_n \wedge \langle b \rangle (\varphi_n \wedge \mathbf{b})) \quad (17)$$

The equivalence (6) now follows by combining (14), (15) and (17).

Since φ_{n+1} is by definition $\langle \varphi_n \rangle \varphi_1$, we use (6) and the induction hypothesis to conclude the equivalence of φ_{n+1} and ψ_{n+1} . \square

We proceed by defining suitable sets of models for the formulae φ_n .

Definition 19. For each $n \geq 1$, the set $\mathcal{W}_{(cd\Gamma)^*}^n$ of $(cd\Gamma)^*$ -alternating words over the alphabet $\{a, b, c, d\}$ is defined in the following way:

- $\mathcal{W}_{(cd\Gamma)^*}^1 = \{cda, cdb\}$;
- $\mathcal{W}_{(cd\Gamma)^*}^n = \{cdaw \mid w \in \mathcal{W}_{(cd\Gamma)^*}^{n-1}\} \cup \{cdbw \mid w \in \mathcal{W}_{(cd\Gamma)^*}^{n-1}\}$, for $n > 1$.

So $(cd\Gamma)^*$ -alternating words in $\mathcal{W}_{(cd\Gamma)^*}^n$ contains expressions from $(cd\Gamma)^n$, i.e., n concatenations of strings from $\{acd, bcd\}$.

Definition 20 (\mathbb{E}^n and \mathbb{F}^n). The two sets of pointed model \mathbb{E}^n , \mathbb{F}^n are constructed by induction on n . (See also Figs. 7 and 8.)

- $\mathbb{E}^1 = \{(\mathcal{E}_w^1, \dot{x}_1) \mid w \in \mathcal{W}_{(cd\Gamma)^*}^1\}$ and $\mathbb{F}^1 = \{(\mathcal{E}_w^1, \dot{x}_1) \mid w \in \mathcal{W}_{(cd\Gamma)^*}^1\}$ where $\mathcal{E}_w^1 = \{E^1, R_w^1, V_w^1\}$ is such that (see also Fig. 7):
 - $E^1 = \{\dot{x}_1, \dot{x}_1, \dot{y}_1, \dot{y}_1, \dot{z}_1, \dot{z}_1, \dot{z}_0, \dot{z}_0\}$.
 - Now assume $w = cdi$, with $i \in \{a, b\}$, then
 - $R_w^1(i) = Cl(\{(\dot{z}_1, \dot{z}_0), (\dot{z}_1, \dot{z}_0), (\dot{y}_1, \dot{y}_1), (\dot{x}_1, \dot{x}_1)\})$;
 - $R_w^1(c) = Cl(\{(\dot{z}_0, \dot{z}_0), (\dot{z}_1, \dot{z}_1), (\dot{x}_1, \dot{y}_1), (\dot{x}_1, \dot{y}_1)\})$;
 - $R_w^1(d) = Cl(\{(\dot{z}_0, \dot{z}_0), (\dot{z}_1, \dot{y}_1), (\dot{z}_1, \dot{y}_1), (\dot{x}_1, \dot{x}_1)\})$;
 - If $j \notin \{i, c, d\}$, then $R_w^1(j) = Cl(\{(\dot{z}_0, \dot{z}_0), (\dot{z}_1, \dot{z}_1), (\dot{y}_1, \dot{y}_1), (\dot{x}_1, \dot{x}_1)\})$;
 - $V_w^1(\mathbf{c}) = \{\dot{y}_1, \dot{y}_1, \dot{z}_1, \dot{z}_1\}$, $V_w^1(\mathbf{d}) = \{\dot{z}_1, \dot{z}_1\}$ and $V_w^1(\mathbf{i}) = \{\dot{z}_0\}$ $V_w^1(\mathbf{p}) = \emptyset$ for any $\mathbf{p} \in A$ such that $\mathbf{p} \notin \{\mathbf{i}, \mathbf{c}, \mathbf{d}\}$.
- Now assume $n > 1$, then $\mathbb{E}^n = \{(\mathcal{E}_w^n, \dot{x}_n) \mid w \in \mathcal{W}_{(cd\Gamma)^*}^n\}$ and $\mathbb{F}^n = \{(\mathcal{E}_w^n, \dot{x}_n) \mid w \in \mathcal{W}_{(cd\Gamma)^*}^n\}$ where $\mathcal{E}_w^n = \{E^n, R_w^n, V_w^n\}$ is such that (see also Fig. 8):
 - $E^n = E^{n-1} \cup \{\dot{x}_n, \dot{x}_n, \dot{y}_n, \dot{y}_n, \dot{z}_n, \dot{z}_n\}$;
 - R_w^n is defined as follows. Since $w \in \mathcal{W}_{(cd\Gamma)^*}^n$, it is of the form $cdjw'$, where $j \in \{a, b\}$, and $w' \in \mathcal{W}_{(cd\Gamma)^*}^{n-1}$.
 - * Then $R_w^n(j) = Cl((R_{w'}^n(j) \setminus \{(\dot{x}_{n-1}, \dot{x}_{n-1}), (\dot{x}_{n-1}, \dot{x}_{n-1})\}) \cup R_j)$, where $R_j = \{(\dot{z}_n, \dot{x}_{n-1}), (\dot{z}_n, \dot{x}_{n-1}), (\dot{y}_n, \dot{y}_n), (\dot{x}_n, \dot{x}_n)\}$;
 - * $R_w^n(c) = Cl(R_{w'}^{n-1}(c) \cup R_c)$, where $R_c = \{(\dot{z}_{n+1}, \dot{z}_{n+1}), (\dot{x}_{n+1}, \dot{y}_{n+1}), (\dot{x}_{n+1}, \dot{y}_{n+1})\}$;
 - * $R_w^n(d) = Cl(R_{w'}^{n-1}(d) \cup R_d)$, where $R_d = \{(\dot{y}_n, \dot{z}_n), (\dot{y}_n, \dot{z}_n), (\dot{x}_n, \dot{x}_n)\}$;
 - * For any $i \notin \{j, c, d\}$, $R_w^n(i) = Cl(R_{w'}^{n-1}(i) \cup R_i)$, where $R_i = \{(\dot{x}_n, \dot{x}_n), (\dot{y}_n, \dot{y}_n), (\dot{z}_n, \dot{z}_n)\}$.
 - $V_w^n(\mathbf{c}) = V_{w'}^{n-1}(\mathbf{c}) \cup \{\dot{y}_n, \dot{y}_n, \dot{z}_n, \dot{z}_n\}$, $V_w^n(\mathbf{d}) = V_{w'}^{n-1}(\mathbf{d}) \cup \{\dot{z}_n, \dot{z}_n\}$, $V_w^n(\mathbf{j}) = V_{w'}^{n-1}(\mathbf{j}) \cup \{\dot{x}_{n-1}, \dot{x}_{n-1}\}$; if $\mathbf{i} \in \{\mathbf{a}, \mathbf{b}\} \setminus \{\mathbf{j}\}$, then $V_w^n(\mathbf{i}) = V_{w'}^{n-1}(\mathbf{i})$; $V_w^n(\mathbf{p}) = \emptyset$ for any $\mathbf{p} \in P$ such that $\mathbf{p} \notin \{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}\}$.

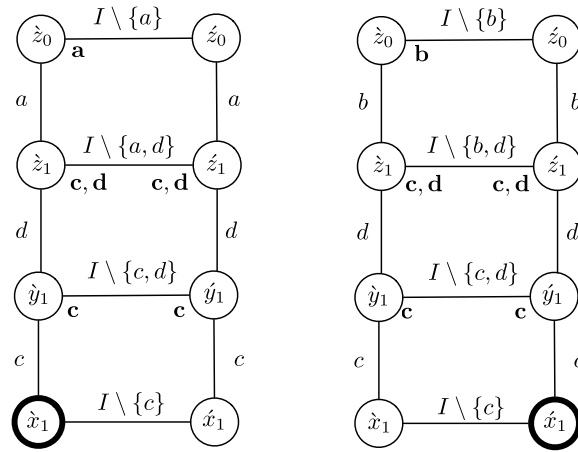


Fig. 7. The models $(\mathcal{E}_{cda}^1, \dot{x}_1)$ (left) and $(\mathcal{E}_{cdb}^1, \dot{x}_1)$ (right).

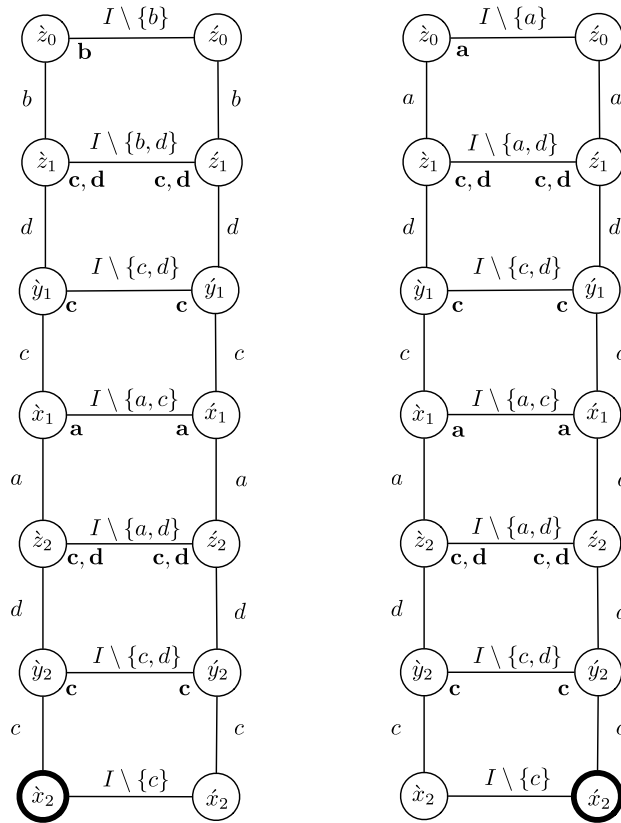


Fig. 8. The models $(\mathcal{A}_{cdacdb}^1, \dot{x}_2)$ (left) and $(\mathcal{A}_{cdacda}^1, \dot{x}_2)$ (right).

So, again, models $(\mathcal{E}_w^n, \dot{x})$ and $(\mathcal{E}_w^n, \dot{x})$ are models with the shape of a ladder: there is a path labelled with the indices in w from \dot{x}_n , the left foot of the ladder, to \dot{z}_0 the left top, labelled with indices according to w . There is an identical path from \dot{x}_n to \dot{z}_0 . From every \dot{x}_i and \dot{x}_i , there is a c -step ‘up’, from every \dot{y}_i and \dot{y}_i there is a d -step ‘up’, and in every \dot{z}_i and \dot{z}_i there is a step ‘up’ labelled either a or b ($i \geq 1$). Two points on a rung \dot{p}_i and \dot{p}_i are labelled with all indices i such that there is no step ‘up’ or ‘down’ in either \dot{p}_i or \dot{p}_i . The atom \mathbf{d} is true in all points \dot{z}_i and \dot{z}_i with $i \geq 1$, atom \mathbf{c} is true in all $\dot{y}_i, \dot{y}_i, \dot{z}_i, \dot{z}_i$ ($i \geq i$) and, finally, for $\mathbf{j} \in \{\mathbf{a}, \mathbf{b}\}$, this atom \mathbf{j} is true in exactly those points p on the left or right support, if p can be reached from a point q by going one step ‘up’ using the label j . In general, it is the case, for any $\mathbf{j} \in \{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}\}$, that if it is possible to reach a point p from q by going ‘up’ using an edge labelled j , then \mathbf{j} is true in p (and on top of that, \mathbf{c} is also true when \mathbf{d} is true).

We have again the following.

Observation 3. For every n , all models in $\mathbb{E}^n \cup \mathbb{F}^n$ are **S5**-models with the property that, for any two points p and p' in the model, and for any $i \in I$: if $p \neq p'$ and $p R_i p'$, then either (p, p') or (p', p) was explicitly added to R_i in the definition of the model (i.e., they are not added because of the transitive closure).

Proposition 6. For all $n \geq 1$, it is true that $\mathbb{E}^n \models \varphi_n$ and $\mathbb{F}^n \models \neg\varphi_n$.

Proof. The proof proceeds along the lines of the proof of Proposition 4, i.e., we prove by induction on j that for every model $\mathcal{E}_w^n = \{E^n, R_w^n, V_w^n\}$ we have

B: for every $1 \leq j \leq n$, it is true that

B1: if $j < i \leq n$, then for all $p_i \in \{\dot{x}_i, \dot{x}_i, \dot{y}_i, \dot{y}_i, \dot{z}_i, \dot{z}_i\}$, then $(\mathcal{E}_w^n, p_i) \models \varphi_j$, and

B2: if $j = i$, then $(\mathcal{E}_w^n, \dot{x}_i) \models \varphi_j$, whereas $(\mathcal{E}_w^n, \dot{x}_i) \models \neg\varphi_j$.

We assume that $1 < n$. The case $n = 1$ is covered in the base case below which we prove in detail. The induction step is left to the reader.

Base case

Let $j = 1$. We are going to argue that $\varphi_1 = \langle c \rangle (\mathbf{c} \wedge \langle d \rangle (\mathbf{d} \wedge (\langle a \rangle \mathbf{a} \vee \langle b \rangle \mathbf{b})))$ is true in any $(\mathcal{E}_w^n, \dot{x}_i)$ and $(\mathcal{E}_w^n, \dot{y}_i)$ with $i > 1$ as long as \dot{y}_i, \dot{z}_i and \dot{x}_{i-1} are present, and it is true in $(\mathcal{E}_w^n, \dot{z}_i)$ as long as \dot{z}_i and \dot{x}_{i-1} are present. As a special case, φ_1 is true in \dot{x}_1 , since we can make a c -step to \dot{y}_1 (where \mathbf{c} is true), from where there is a d -step to \dot{z}_1 (where \mathbf{d} is true), and from there, depending on w , either an a or a b -step to \dot{z}_0 , where either \mathbf{a} or \mathbf{b} is true. For the right support, φ_1 is true in any $(\mathcal{E}_w^n, \dot{x}_i)$ and $(\mathcal{E}_w^n, \dot{y}_i)$ with $i > 1$ as long as \dot{y}_i, \dot{z}_i and \dot{x}_{i-1} are present, and it is true in $(\mathcal{E}_w^n, \dot{z}_i)$ as long as \dot{z}_i and \dot{x}_{i-1} are present. As a special case, φ_1 is false in \dot{x}_1 : to make a c -step to a point where \mathbf{c} is true, we must go to \dot{y}_1 , and from there, to make a d -step to a point where \mathbf{d} is true, we must go to \dot{z}_1 . However, from there, there is neither an a -step to a point where \mathbf{a} is true, nor a b -step to a point where \mathbf{b} holds. The following proves this argument a little more formally:

B1: Let $j < i$. For every $1 < k \leq n$, we have:

1. $\dot{x}_k R_c \dot{y}_k$ and $\dot{x}_k R_c \dot{y}_k$, and $\dot{y}_k, \dot{y}_k \in V_w^n(\mathbf{c})$;
2. $\dot{y}_k R_d \dot{z}_k$ and $\dot{y}_k R_d \dot{z}_k$, and $\dot{z}_k, \dot{z}_k \in V_w^n(\mathbf{d})$;
3. There are $l \in \{a, b\}$ and $\mathbf{l} \in \{\mathbf{a}, \mathbf{b}\}$, such that $\dot{z}_k R_l \dot{x}_{k-1}$ and $\dot{z}_k R_l \dot{x}_{k-1}$, and $\dot{x}_{k-1}, \dot{x}_{k-1} \in V_w^n(\mathbf{l})$;
Therefore, $(\mathcal{E}_w^n, \dot{x}_k) \models \langle c \rangle (\mathbf{c} \wedge \langle d \rangle (\mathbf{d} \wedge (\langle a \rangle \mathbf{a} \vee \langle b \rangle \mathbf{b})))$ and the same is true about $(\mathcal{E}_w^n, \dot{x}_k)$, i.e., $(\mathcal{E}_w^n, \dot{x}_k) \models \langle c \rangle (\mathbf{c} \wedge \langle d \rangle (\mathbf{d} \wedge (\langle a \rangle \mathbf{a} \vee \langle b \rangle \mathbf{b})))$.

The proof of the statement for \dot{y}_k and \dot{y}_k is similar. Indeed, given the shape of \mathcal{E}_w^n , the following are true:

1. $\dot{y}_k R_c \dot{y}_k$ and $\dot{y}_k R_c \dot{y}_k$, and $\dot{y}_k, \dot{y}_k \in V_w^n(\mathbf{c})$;
2. $\dot{y}_k R_d \dot{z}_k$ and $\dot{y}_k R_d \dot{z}_k$, and $\dot{z}_k, \dot{z}_k \in V_w^n(\mathbf{d})$;
3. There are $l \in \{a, b\}$ and $\mathbf{l} \in \{\mathbf{a}, \mathbf{b}\}$, such that $\dot{z}_k R_l \dot{x}_{k-1}$ and $\dot{z}_k R_l \dot{x}_{k-1}$, and $\dot{x}_{k-1}, \dot{x}_{k-1} \in V_w^n(\mathbf{l})$;
Therefore, $(\mathcal{E}_w^n, \dot{y}_k) \models \varphi_1$ and $(\mathcal{E}_w^n, \dot{x}_k) \models \varphi_1$.

Similarly,

1. $\dot{z}_k R_c \dot{z}_k$ and $\dot{z}_k R_c \dot{z}_k$, and $\dot{z}_k, \dot{z}_k \in V_w^n(\mathbf{c}) \cup V_w^n(\mathbf{d})$;
2. There are $l \in \{a, b\}$ and $\mathbf{l} \in \{\mathbf{a}, \mathbf{b}\}$, such that $\dot{z}_k R_l \dot{x}_{k-1}$ and $\dot{z}_k R_l \dot{x}_{k-1}$, and $\dot{x}_{k-1}, \dot{x}_{k-1} \in V_w^n(\mathbf{l})$;
Hence, $(\mathcal{E}_w^n, \dot{z}_k) \models \varphi_1$ and $(\mathcal{E}_w^n, \dot{z}_k) \models \varphi_1$.

B2: To complete the base case, we have to prove that $(\mathcal{E}_w^n, \dot{x}_1) \models \varphi_1$ whereas $(\mathcal{E}_w^n, \dot{x}_1) \models \neg\varphi_1$. Indeed, we have

1. $\dot{x}_1 R_c \dot{y}_1$ and $\dot{y}_1 \in V_w^n(\mathbf{c})$;
2. $\dot{y}_1 R_d \dot{z}_1$ and $\dot{z}_1 \in V_w^n(\mathbf{d})$;
3. There are $l \in \{a, b\}$ and $\mathbf{l} \in \{\mathbf{a}, \mathbf{b}\}$, such that $\dot{z}_1 R_l \dot{z}_0$ and $\dot{z}_0 \in V_w^n(\mathbf{l})$;
Therefore, $(\mathcal{E}_w^n, \dot{x}_1) \models \langle c \rangle (\mathbf{c} \wedge \langle d \rangle (\mathbf{d} \wedge (\langle a \rangle \mathbf{a} \vee \langle b \rangle \mathbf{b})))$ and, thus, $(\mathcal{E}_w^n, \dot{x}_1) \models \varphi_1$.

It is easy to see that $\neg\varphi_1$ is equivalent to $[c](\neg\mathbf{c} \vee [d](\neg\mathbf{d} \vee ([a]\neg\mathbf{a} \wedge [b]\neg\mathbf{b})))$.

Again, given the shape of the model \mathcal{E}_w^n , it is easy to see that the following are true

1. $(\mathcal{E}_w^n, \dot{x}_1) \models \neg\mathbf{c}$;
2. $(\mathcal{E}_w^n, \dot{z}_1) \models ([a]\neg\mathbf{a} \wedge [b]\neg\mathbf{b})$;
3. $(\mathcal{E}_w^n, \dot{y}_1) \models \neg\mathbf{d}$;
4. $\dot{y}_1 R_d \dot{z}_1$, and there is no point p that is a d -successor of either \dot{y}_1 or \dot{z}_1 and such that $(\mathcal{E}_w^n, p) \models (\mathbf{d} \wedge (\langle a \rangle \mathbf{a} \vee \langle b \rangle \mathbf{b}))$.
Therefore, we see that $(\mathcal{E}_w^n, \dot{y}_1) \models [d](\neg\mathbf{d} \vee ([a]\neg\mathbf{a} \wedge [b]\neg\mathbf{b}))$.

Given all the items above, the fact that $\dot{x}_1 R_c \dot{y}_1$ and there is no point p such that p is a c -successor of either \dot{x}_1 or \dot{y}_1 and $(\mathcal{E}_w^n, p) \models \langle c \rangle (\mathbf{c} \wedge \langle d \rangle (\mathbf{d} \wedge (\langle a \rangle \mathbf{a} \vee \langle b \rangle \mathbf{b})))$, we see that $(\mathcal{E}_w^n, \dot{x}_1) \models [c](\neg\mathbf{c} \vee [d](\neg\mathbf{d} \vee ([a]\neg\mathbf{a} \wedge [b]\neg\mathbf{b})))$ and, thus, $(\mathcal{E}_w^n, \dot{x}_1) \models \neg\varphi_1$.

Induction step

Suppose that $j + 1 < i \leq n$ and let us assume that the statement **B** is true for j . So, we have

Table 4
Nodes.

$\langle \mathbb{H}_1 \cup \{(\mathcal{H}_w^n, p_i)\} \circ \mathbb{H}_2 \cup \{(\mathcal{H}_w^n, q_i)\} \rangle$	$\langle \mathbb{H}_2 \cup \{(\mathcal{H}_w^n, q_i)\} \circ \mathbb{H}_1 \cup \{(\mathcal{H}_w^n, p_i)\} \rangle$
$\langle \mathbb{H}_1 \cup \{(\mathcal{H}_w^n, \hat{x}_i)\} \circ \mathbb{H}_2 \cup \{(\mathcal{H}_w^n, \hat{x}_i)\} \rangle$	$\langle \mathbb{H}_2 \cup \{(\mathcal{H}_w^n, \hat{x}_i)\} \circ \mathbb{H}_1 \cup \{(\mathcal{H}_w^n, \hat{x}_i)\} \rangle$
$\langle \mathbb{H}_1 \cup \{(\mathcal{H}_w^n, \hat{y}_i)\} \circ \mathbb{H}_2 \cup \{(\mathcal{H}_w^n, \hat{y}_i)\} \rangle$	$\langle \mathbb{H}_2 \cup \{(\mathcal{H}_w^n, \hat{y}_i)\} \circ \mathbb{H}_1 \cup \{(\mathcal{H}_w^n, \hat{y}_i)\} \rangle$
$\langle \mathbb{H}_1 \cup \{(\mathcal{H}_w^n, \hat{z}_i)\} \circ \mathbb{H}_2 \cup \{(\mathcal{H}_w^n, \hat{z}_i)\} \rangle$	$\langle \mathbb{H}_2 \cup \{(\mathcal{H}_w^n, \hat{z}_i)\} \circ \mathbb{H}_1 \cup \{(\mathcal{H}_w^n, \hat{z}_i)\} \rangle$

B1: If $j < i \leq n$, then for all $p_i \in \{\hat{x}_i, \hat{x}_i, \hat{y}_i, \hat{y}_i, \hat{z}_i, \hat{z}_i\}$, then $(\mathcal{E}_w^n, p_i) \models \varphi_j$, and

B2: If $j = i$, then $(\mathcal{E}_w^n, \hat{x}_i) \models \varphi_j$, whereas $(\mathcal{E}_w^n, \hat{x}_i) \models \neg \varphi_j$.

To prove **B1** for $j + 1$, assume $j + 1 < i \leq n$. By definition, $\varphi_{j+1} = \langle \varphi_j \rangle \varphi_1$, which using (5), is equivalent to $\langle \varphi_j \rangle \langle c \rangle (\mathbf{c} \wedge \langle d \rangle (\mathbf{d} \wedge \langle a \rangle \mathbf{a} \vee \langle b \rangle \mathbf{b}))$. Recall that $\langle \varphi_j \rangle \varphi_1$ is shorthand for $\neg[\varphi_j] \neg \varphi_1$, so that

$$(\mathcal{E}_w^n, p) \models \langle \varphi_j \rangle \varphi_1 \quad \text{iff} \quad (\mathcal{E}_w^n, p) \models \varphi_j \quad \text{and} \quad (\mathcal{E}_w^n|_{\varphi_j}, p) \models \varphi_1 \quad (18)$$

From the induction hypothesis, we know that φ_j is true in the following points of the left support of \mathcal{E}_w^n :

$$\dot{P} = \{\hat{x}_n, \hat{y}_n, \hat{z}_n, \dots, \hat{x}_{j+1}, \hat{y}_{j+1}, \hat{z}_{j+1}, \hat{x}_j\}$$

and those at the right support:

$$\dot{P} = \{\hat{x}_n, \hat{y}_n, \hat{z}_n, \dots, \hat{x}_{j+1}, \hat{y}_{j+1}, \hat{z}_{j+1}\}$$

(Crucially, \hat{x}_j is not present in \dot{P} .) Thus, we have

$$(\mathcal{E}_w^n, p) \models \varphi_j \quad \text{iff} \quad p \in \dot{P} \cup \dot{P} \quad (19)$$

This implies that if φ_j is announced, the points $\dot{P} \cup \dot{P}$ will re-appear in the updated model $\mathcal{E}_w^n|_{\varphi_j}$, and so will their connections. It is easy to verify, given the construction of our model, that $\varphi_1 = \langle c \rangle (\mathbf{c} \wedge \langle d \rangle (\mathbf{d} \wedge \langle a \rangle \mathbf{a} \vee \langle b \rangle \mathbf{b}))$ holds in $\mathcal{E}_w^n|_{\varphi_j}$ in any point \dot{p} from $\{\hat{x}_i, \hat{y}_i, \hat{z}_i\}$ iff all of \hat{y}_i, \hat{z}_i and \hat{x}_{i-1} are present. Likewise, φ_1 holds in $\mathcal{E}_w^n|_{\varphi_j}$ in any point \dot{p} from $\{\hat{x}_i, \hat{y}_i, \hat{z}_i\}$ iff all of \hat{y}_i, \hat{z}_i and \hat{x}_{i-1} are present. Together, this shows that $\langle \varphi_{j+1} \rangle \varphi_1$ is true in any point in the model $\mathcal{E}_w^n|_{\varphi_j}$ from the set

$$\dot{Q} = \{\hat{x}_n, \hat{y}_n, \hat{z}_n, \dots, \hat{x}_{j+2}, \hat{y}_{j+2}, \hat{z}_{j+2}, \hat{x}_{j+1}\}$$

together with

$$\dot{Q} = \{\hat{x}_n, \hat{y}_n, \hat{z}_n, \dots, \hat{x}_{j+2}, \hat{y}_{j+2}, \hat{z}_{j+2}\}$$

(Point $\hat{x}_{j+1} \notin \dot{Q}$ because, in order for φ_1 to be true in it, it needs \hat{x}_j , which is not present in \dot{P} , and hence not in $\mathcal{E}_w^n|_{\varphi_j}$.) Hence

$$(\mathcal{E}_w^n|_{\varphi_j}, p) \models \varphi_1 \quad \text{iff} \quad p \in \dot{Q} \cup \dot{Q} \quad (20)$$

Using (18), (19) and (20), we see that

$$(\mathcal{E}_w^n, p) \models \langle \varphi_j \rangle \varphi_1 \quad \text{iff} \quad p \in (\dot{P} \cap \dot{Q}) \cup (\dot{P} \cap \dot{Q}) = (\dot{Q} \cap \dot{Q}) \quad (21)$$

From this, and the fact that $\varphi_{j+1} = \langle \varphi_j \rangle \varphi_1$, we obtain **B1** for $j + 1 < i \leq n$, i.e., for all $p_i \in \{\hat{x}_i, \hat{x}_i, \hat{y}_i, \hat{y}_i, \hat{z}_i, \hat{z}_i\}$, we have $(\mathcal{E}_w^n, p_i) \models \varphi_{j+1}$. And we also obtain (since $\hat{x}_{j+1} \in \dot{Q}$) that $(\mathcal{E}_w^n, \hat{x}_{j+1}) \models \varphi_{j+1}$ but (since $\hat{x}_{j+1} \notin \dot{Q}$) that $(\mathcal{E}_w^n, \hat{x}_{j+1}) \models \neg \varphi_{j+1}$, i.e., **B2** holds for $j + 1 = i$.

$$\varphi_j \wedge \langle c \rangle (\varphi_j \wedge \mathbf{c} \wedge \langle d \rangle (\varphi_j \wedge \mathbf{d} \wedge (\langle a \rangle (\varphi_{i-1} \wedge \mathbf{a}) \vee \langle b \rangle (\varphi_{i-1} \wedge \mathbf{b})))) \quad \square$$

Propositions 4 and 6 imply that Spoiler can win any one of the games starting at $\langle \mathbb{A}^n \circ \mathbb{B}^n \rangle$ or $\langle \mathbb{C}^n \circ \mathbb{D}^n \rangle$ or $\langle \mathbb{E}^n \circ \mathbb{F}^n \rangle$.

3.2.2. Number of moves needed for FSGs on our models

We proceed to proving that winning any of the games starting at $\langle \mathbb{A}^n \circ \mathbb{B}^n \rangle$ or $\langle \mathbb{C}^n \circ \mathbb{D}^n \rangle$ or $\langle \mathbb{E}^n \circ \mathbb{F}^n \rangle$ cannot be done in less than 2^n moves. From now on, \mathcal{H}_w^n will denote some arbitrary but fixed model from the set $\{\mathcal{A}_w^n, \mathcal{C}_w^n, \mathcal{E}_w^n\}$. The next property of the models in $\mathbb{A}^n, \mathbb{B}^n, \mathbb{C}^n, \mathbb{D}^n, \mathbb{E}^n$, and \mathbb{F}^n is essential.

Let us consider the nodes in Table 4.

Proposition 7. Let \mathcal{H}_w^n range over $\{\mathcal{A}_w^n, \mathcal{C}_w^n, \mathcal{E}_w^n\}$, and let the pairs (\dot{p}_i, \dot{p}_i) range over $\{(\hat{x}_i, \hat{x}_i), (\hat{y}_i, \hat{y}_i)\}$ ($1 \leq i \leq n$) and $\{(\hat{z}_i, \hat{z}_i)\}$ ($i \geq 2$ in case of $\mathcal{A}_w^n, \mathcal{C}_w^n$ and $i \geq 0$ in case of \mathcal{E}_w^n). Let $r \in I$. For any game tree T that contains at least one of the nodes $\langle \mathbb{H}_1 \cup \{(\mathcal{H}_w^n, \dot{p}_i)\} \circ \mathbb{H}_2 \cup \{(\mathcal{H}_w^n, \dot{p}_i)\} \rangle$ or $\langle \mathbb{H}_2 \cup \{(\mathcal{H}_w^n, \dot{p}_i)\} \circ \mathbb{H}_1 \cup \{(\mathcal{H}_w^n, \dot{p}_i)\} \rangle$, if Spoiler plays an $[r]$ -move while \dot{p}_i is an r -successor of \dot{p}_i , then T cannot be closed (i.e., Spoiler loses the formula size game).

Proof.

This follows immediately from the definition of the $[r]$ -move, which if played at a node of this shape, and $\mathcal{H}_w^n \in \{\mathcal{A}_w^n, \mathcal{E}_w^n\}$, will result in adding to the game tree one of the following nodes

- $\langle \mathbb{H}'_1 \cup \{(\mathcal{H}_w^n, \dot{p}_i)\} \circ \mathbb{H}'_2 \cup \{(\mathcal{H}_w^n, \dot{p}_i)\} \rangle$ or $\langle \mathbb{H}'_1 \cup \{(\mathcal{H}_w^n, \dot{p}_i)\} \circ \mathbb{H}'_2 \cup \{(\mathcal{H}_w^n, \dot{p}_i)\} \rangle$;
- $\langle \mathbb{H}'_2 \cup \{(\mathcal{H}_w^n, \dot{p}_i)\} \circ \mathbb{H}'_1 \cup \{(\mathcal{H}_w^n, \dot{p}_i)\} \rangle$ or $\langle \mathbb{H}'_2 \cup \{(\mathcal{H}_w^n, \dot{p}_i)\} \circ \mathbb{H}'_1 \cup \{(\mathcal{H}_w^n, \dot{p}_i)\} \rangle$;

or, if \mathcal{H}_w^n is \mathcal{C}_w^n and $\bar{m}_i \in \{\bar{y}_i, \bar{z}_i, \bar{x}_i\}$, this leads to either adding one node of the forms above or

- $\langle \mathbb{H}'_1 \cup \{(\mathcal{H}_w^n, \bar{m}_i)\} \circ \mathbb{H}'_2 \cup \{(\mathcal{H}_w^n, \bar{m}_i)\} \rangle$;
- $\langle \mathbb{H}'_2 \cup \{(\mathcal{H}_w^n, \bar{m}_i)\} \circ \mathbb{H}'_1 \cup \{(\mathcal{H}_w^n, \bar{m}_i)\} \rangle$.

Given the fact that in any one of these nodes we have two bisimilar models, one on the left and one on the right, we see that Spoiler cannot close T . \square

We now define the notion of a path in a model, and, more specially, of an ℓ -path in our ladder-like models: intuitively, an ℓ -path is a sequence of steps, which leads one from the left bottom point to the left top point, and in which only points on the left support of the model are visited. In other words, the individual steps in the sequence can only be 'up', 'down', or 'stay in the current point'. However, if one stays in the current point using the index i , then it would not have been possible to take a horizontal step using i .

Definition 21 (Paths and ℓ -paths).

- Let $\mathcal{M} = \langle M, R, V \rangle$ be a model over a signature $\langle A, I \rangle$. A path π in \mathcal{M} is a sequence π of indices $\pi_1, \pi_2, \dots, \pi_k$ such that there is a sequence of points p_1, \dots, p_{k+1} in M , for which $p_1 = p$, $p_{k+1} = q$, and for any pair of points (p_i, p_{i+1}) ($i \leq k$), we have $p_i R_{\pi_i} p_{i+1}$. We also say that this path runs from p_1 to p_{k+1} . Moreover, the sequence $\varepsilon = p_1 \pi_1 p_2 \dots p_k \pi_k p_{k+1}$ is called an extended path for π . Note that a path π can give rise to several extended paths ε for π , but an extended path over π uniquely determines the path π . If we refer to an (extended) path in a pointed model (\mathcal{M}, p) , it is always a path that runs from p .
- Let $w \in \mathcal{W}_{\Gamma(\text{cd}\Gamma)^*}^n \cup \mathcal{W}_{(\text{cd}\Gamma)^*}^n$, and the pointed model $(\mathcal{M}_w^n, \dot{x}_n) \in \{(\mathcal{A}_w^n, \dot{x}_n), (\mathcal{C}_w^n, \dot{x}_n), (\mathcal{E}_w^n, \dot{x}_n)\}$. We define the left support \dot{P}_n of $(\mathcal{M}_w^n, \dot{x}_n)$ to consist of all the points $\{\dot{x}_n, \dot{y}_n, \dot{z}_n, \dots, \dot{x}_2, \dot{y}_2, \dot{z}_2, \dot{x}_1, \dot{y}_1\}$ in case $(\mathcal{M}_w^n, \dot{x}_n) \in \{(\mathcal{A}_w^n, \dot{x}_n), (\mathcal{C}_w^n, \dot{x}_n)\}$, and $\dot{P}_n = \{\dot{x}_n, \dot{y}_n, \dot{z}_n, \dots, \dot{x}_1, \dot{y}_1, \dot{z}_1, \dot{z}_0\}$ in the case that $(\mathcal{M}_w^n, \dot{x}_n) = (\mathcal{E}_w^n, \dot{x}_n)$. In the former case, we define $\text{top}(\dot{P}) = \dot{y}_1$, and in the latter case, $\text{top}(\dot{P}) = \dot{z}_0$. In both cases, $\text{bot}(\dot{P}) = \dot{x}_n$. Given $(\mathcal{M}_w^n, \dot{x}_n)$, an ℓ -path π over w is a sequence of indices $i = \pi_1, \pi_2, \dots, \pi_k$ such that there is a sequence of points $\dot{p}_1, \dot{p}_2, \dots, \dot{p}_{k+1}$ satisfying $\{\dot{p}_1, \dot{p}_2, \dots, \dot{p}_{k+1}\} = \dot{P}$ with $\dot{p}_1 = \text{bot}(\dot{P})$ and $\dot{p}_{k+1} = \text{top}(\dot{P})$, and for every pair of points $(\dot{p}_i, \dot{p}_{i+1})$, we have $\dot{p}_i R_{\pi_i} \dot{p}_{i+1}$. Moreover, it is *not* the case, for any i , that $\dot{p}_i R_{\pi_i} \dot{p}_i$. In this case, $\varepsilon = \dot{p}_1 \pi_1 \dot{p}_2 \dots \dot{p}_k \pi_k \dot{p}_{k+1}$ is called an extended ℓ -path for π .

Note that a path can connect several points (in particular, since all the accessibility relations are reflexive, every path connects every point with itself). The following lemma formalises the fact that in all our models, if an ℓ -path contains a substring xx , then a point must be visited twice. Similarly for ℓ -paths containing a substring xyx .

Lemma 4. Take any $w \in \mathcal{W}_{\Gamma(\text{cd}\Gamma)^*}^n \cup \mathcal{W}_{(\text{cd}\Gamma)^*}^n$, and let $\pi = \pi_1, \pi_2, \dots, \pi_k$ be an ℓ -path over w . If π contains a substring xx or xyx , then any extended ℓ -path $\dot{p}_1 \pi_1 \dot{p}_2 \dots \dot{p}_k \pi_k \dot{p}_{k+1}$ must have a j with $\pi_j = x$ and either $\dot{p}_j = \dot{p}_{j+1}$ or $\dot{p}_j = \dot{p}_{j+2}$.

Proof. An ℓ -path is a sequence of indices that takes one from the bottom of the left support \dot{x}_n to the top on that support, only visiting points in \dot{P} and never using an index, at \dot{p} , that is an index connecting it to \dot{p} . The definition of our models is such that, when traversing from bottom to top, every two different occurrences of an index x at the left support, are always separated by at least two other indices (every two different occurrences of a are separated by a c and a d , every two different occurrences of b are separated by c and d , every two occurrences of c are separated by a d and an index from $\{a, b\}$, and every two occurrences of b are separated by c and an index from $\{a, b\}$). From this the lemma follows directly. More specifically, we have the following cases:

1. Suppose $\pi_m = \pi_{m+1} = x$. Then the extended ℓ -path contains one of the following sequences:
 - (a) $\dot{p}_m x \dot{p}_m$ (the path stutters in \dot{p}_m , the claim is true with $m = j$);
 - (b) $\dot{p}_m x \dot{p}_{m+1} x \dot{p}_m$ with $\dot{p}_m \neq \dot{p}_{m+1}$ (the path re-visits \dot{p}_m , the claim is true with $m = j$);
 - (c) $\dot{p}_m x \dot{p}_{m+1} x \dot{p}_{m+1}$ with $\dot{p}_m \neq \dot{p}_{m+1}$ (the path stutters in \dot{p}_{m+1} , the claim is true for $j = m + 1$);
2. Suppose $\pi_m \pi_{m+1} \pi_{m+2} = xyx$ with $x \neq y$. Then the extended ℓ -path contains one of the following sequences:
 - (a) $\dot{p}_m x \dot{p}_m$ (the path stutters in \dot{p}_m , the claim is true with $m = j$);
 - (b) $\dot{p}_m x \dot{p}_{m+1} y \dot{p}_m$ with $\dot{p}_m \neq \dot{p}_{m+1}$ (the path re-visits \dot{p}_m , the claim is true with $m = j$);

(c) $\dot{p}_m \times \dot{p}_{m+1} y \dot{p}_{m+1}$ with $\dot{p}_m \neq \dot{p}_{m+1}$ (the path stutters in \dot{p}_{m+1} , the claim is true for $j = m + 1$);

These are all the possibilities in this case, in particular, it is impossible to have a sequence $\dot{p}_m \times \dot{p}_{m+1} y \dot{p}_{m+2} \times$ with $\dot{p}_m \neq \dot{p}_{m+1}$, $\dot{p}_{m+1} \neq \dot{p}_{m+2}$ and $\dot{p}_m \neq \dot{p}_{m+2}$. \square

We also have the following simple proposition, which says that in any model $\mathcal{H}_w^n \in \mathbb{A}^n \cup \mathbb{B}^n \cup \mathbb{C}^n \cup \mathbb{D}^n \cup \mathbb{E}^n \cup \mathbb{F}^n$, the word w is an ℓ -path.

Proposition 8. Suppose that $w \in \mathcal{W}_{\Gamma(\text{cd}\Gamma)^*}^n$ (or $w \in \mathcal{W}_{(\text{cd}\Gamma)^*}^n$) and the word $w = \text{icdjcd} \dots \text{cdl}$ (or $w = \text{cdicdj} \dots \text{cdl}$), where $i, j, l \in \{a, b\}$. Then the sequence of indices $\pi_w = \text{icdjcd} \dots \text{cdl}$ (or $\pi_w = \text{cdicdj} \dots \text{cdl}$) is an ℓ -path over w in the pointed models $(\mathcal{A}_w^n, \dot{x}_n)$ and $(\mathcal{C}_w^n, \dot{x}_n)$ (in the model $(\mathcal{E}_w^n, \dot{x}_n)$).

Proof. It is obvious that there is a sequence of points, namely $\dot{x}_n, \dot{y}_n, \dot{z}_n, \dots, \dot{x}_2, \dot{y}_2, \dot{z}_2, \dot{x}_1, \dot{y}_1$ (or $\dot{x}_n, \dot{y}_n, \dot{z}_n, \dots, \dot{x}_1, \dot{y}_1, \dot{z}_1, \dot{z}_0$) such that the requirements of Definition 21 are fulfilled. \square

Definition 22 (Canonical ℓ -paths). For any word $w \in \mathcal{W}_{\Gamma(\text{cd}\Gamma)^*}^n \cup \mathcal{W}_{(\text{cd}\Gamma)^*}^n$, the ℓ -path π_w defined in Proposition 8 is called the canonical ℓ -path over w . Let $\pi = \pi_1 \dots \pi_k$ be an ℓ -path over $w \in \mathcal{W}_{\Gamma(\text{cd}\Gamma)^*}^n$ and let p_1, p_2, \dots, p_{k+1} be the points satisfying Definition 21. Recall that $p_1 \pi_1 p_2, \dots, p_k \pi_k p_{k+1}$ is called the extended ℓ -path over w . If $\pi = \pi_w$, then this word is called the extended canonical ℓ -path. Similarly, we define extended ℓ -paths and the extended canonical ℓ -path over $w \in \mathcal{W}_{(\text{cd}\Gamma)^*}^n$.

Note that the extended canonical ℓ -path is unique, i.e., the point p_1 is always \dot{x}_n , the point p_2 is always \dot{y}_n , the point p_3 is always \dot{z}_n etc.

As we said earlier, intuitively, the word w encodes the shortest ℓ -path from \dot{x}_n to \dot{y}_0 in the case of the models $(\mathcal{A}_w^n, \dot{x}_n)$ and $(\mathcal{C}_w^n, \dot{x}_n)$ and the shortest ℓ -path from \dot{x}_n to \dot{z}_0 in the model $(\mathcal{E}_w^n, \dot{x}_n)$. Using Definitions 21 and 22, we can express this more formally as follows.

Proposition 9. If $w \in \mathcal{W}_{\Gamma(\text{cd}\Gamma)^*}^n$, then π_w is the shortest ℓ -path over w in the models $(\mathcal{A}_w^n, \dot{x}_n)$ and $(\mathcal{C}_w^n, \dot{x}_n)$ and, similarly, if $w \in \mathcal{W}_{(\text{cd}\Gamma)^*}^n$, then π_w is the shortest ℓ -path over w in the model $(\mathcal{E}_w^n, \dot{x}_n)$.

Proof. Using Definition 21, it is easy to see that each one of the points $\dot{x}_n, \dot{y}_n, \dot{z}_n, \dots, \dot{x}_2, \dot{y}_2, \dot{z}_2, \dot{x}_1, \dot{y}_1$ (and \dot{z}_0 in the case of $(\mathcal{E}_w^n, \dot{x}_n)$) must appear at least once in any extended ℓ -path over w . Each one of these points appears exactly once in the extended canonical ℓ -path π_w . Hence, there is no shorter ℓ -path over w in the models $(\mathcal{A}_w^n, \dot{x}_n)$ and $(\mathcal{C}_w^n, \dot{x}_n)$ (or in the model $(\mathcal{E}_w^n, \dot{x}_n)$). \square

Lemma 5. Every $T \in \mathcal{T}((\mathcal{H}_w^n, \dot{x}_n) \circ (\mathcal{H}_w^n, \dot{x}_n))$ has a branch B such that $I(B)$ is an ℓ -path over w in $(\mathcal{H}_w^n, \dot{x}_n)$.

Proof. As in the proof of Lemma 3, B is constructed inductively. This time, however, the induction is not on n but it is an inductive procedure for specifying B as a branch consisting of points with certain properties. To present the main idea of the proof in its full generality, we assume that $n > 1$, $w \in \mathcal{W}_{\Gamma(\text{cd}\Gamma)^*}^n$ and \mathcal{H}_w^n is one of the models \mathcal{A}_w^n or \mathcal{C}_w^n . The case $n = 1$ or $w \in \mathcal{W}_{(\text{cd}\Gamma)^*}^n$ and \mathcal{H}_w^n is \mathcal{E}_w^n is completely analogous.

During the construction, certain pointed models (\mathcal{H}_w^n, p) will be declared *marked*. Once a model has been marked, it remains marked until the end of the game. If we have a node η that contains two marked models, (\mathcal{H}_w^n, p) on the left and (\mathcal{H}_w^n, q) on the right, then playing an *or*-move at η will result in adding at least one node η_1 as successor of η that contains the marked (\mathcal{H}_w^n, p) on the left and (\mathcal{H}_w^n, q) on the right. Playing a *not*-move will result in adding a successor of η that contains the same marked models but this time (\mathcal{H}_w^n, q) on the left and (\mathcal{H}_w^n, p) on the right.

Let us construct the desired branch B . Suppose that $w = \text{icdjcd} \dots \text{cdk}$, where $i, j, k \in \{a, b\}$. The first point η_0 of B is the root of T and both models $(\mathcal{H}_w^n, \dot{x}_n)$, and $(\mathcal{H}_w^n, \dot{x}_n)$ are marked. If some number $0 \leq l$ of *not* or *or*-moves were played first, we “follow” the marked models $(\mathcal{H}_w^n, \dot{x}_n)$, and $(\mathcal{H}_w^n, \dot{x}_n)$. The above considerations show that there are nodes η_0, \dots, η_l , that contain them. We add η_0, \dots, η_l to B . Since these models satisfy the same propositional symbols, η_l cannot be closed and, therefore, an $[r]$ -move is played. It follows from Proposition 7, that $r = i$. There are two cases

- $(\mathcal{H}_w^n, \dot{x}_n)$ is on the left and $(\mathcal{H}_w^n, \dot{x}_n)$ is on the right;
- $(\mathcal{H}_w^n, \dot{x}_n)$ is on the right and $(\mathcal{H}_w^n, \dot{x}_n)$ is on the left.

For each one of these cases, there are two possibilities for playing an $[r]$ -move. We consider these two possibilities for the first case. The second case follows by symmetry.

1. Spoiler has chosen $(\mathcal{H}_w^n, \dot{x}_n)$ on the right. This is possible because all the relations in \mathcal{H}_w^n , including i , are reflexive. It follows immediately that one of the chosen models on the left is $(\mathcal{H}_w^n, \dot{x}_n)$. We declare these two models marked and add the node η_{l+1} to B ;

2. Spoiler has chosen $(\mathcal{H}_w^n, \dot{y}_n)$ on the right. This is possible because \dot{y}_n is an i -successor of \dot{x}_n . It follows that one of the chosen models on the left is $(\mathcal{H}_w^n, \dot{y}_n)$. These models are declared marked and we add the node η_{l+1} to B .

Since the marked models in η_{l+1} satisfy the same propositional symbols, η_{l+1} cannot be closed. Again, we “follow” the newly marked models to a node where a $[i]$ -move is played.

In general,

- If a node η_l contains marked models of the form $(\mathcal{H}_w^n, \dot{x}_j)$ on the left and $(\mathcal{H}_w^n, \dot{x}_j)$ on the right (or vice versa) and an $[r]$ -move was played at this node then, using Proposition 7, we see that \dot{x}_j and \dot{x}_j are not r -connected. Hence r is either d or $r \in \{a, b\}$ and \dot{y}_j and \dot{x}_j are r -connected. Therefore, there are not more than three possibilities for the successor node η_{l+1} .
 1. η_{l+1} contains $(\mathcal{H}_w^n, \dot{x}_j)$ on the left and $(\mathcal{H}_w^n, \dot{x}_j)$ on the right (or vice versa) and these models are marked;
 2. η_{l+1} contains $(\mathcal{H}_w^n, \dot{z}_{j+1})$ on the left and $(\mathcal{H}_w^n, \dot{z}_{j+1})$ on the right (or vice versa) and these models are marked;
 3. η_{l+1} contains $(\mathcal{H}_w^n, \dot{y}_j)$ on the left and $(\mathcal{H}_w^n, \dot{y}_j)$ on the right (or vice versa) and these models are marked.
- If a node η_l contains marked models of the form $(\mathcal{H}_w^n, \dot{y}_j)$ on the left and $(\mathcal{H}_w^n, \dot{y}_j)$ on the right (or vice versa) and an $[r]$ -move was played at this node, then Proposition 7 implies that \dot{y}_j and \dot{y}_j are not r -connected. Hence, j is either c or $j \in \{a, b\}$ and \dot{y}_j is a j -successor of \dot{x}_n . As before, there are not more than three possibilities for the successor node η_{l+1} .
 1. η_{l+1} contains $(\mathcal{H}_w^n, \dot{y}_j)$ on the left and $(\mathcal{H}_w^n, \dot{y}_j)$ on the right (or vice versa) and these models are marked;
 2. η_{l+1} contains $(\mathcal{H}_w^n, \dot{z}_j)$ on the left and $(\mathcal{H}_w^n, \dot{z}_j)$ on the right (or vice versa) and these models are marked;
 3. η_{l+1} contains $(\mathcal{H}_w^n, \dot{x}_{j+1})$ on the left and $(\mathcal{H}_w^n, \dot{x}_{j+1})$ on the right (or vice versa) and these models are marked.
- If a node η_l contains marked models of the form $(\mathcal{H}_w^n, \dot{z}_j)$ on the left and $(\mathcal{H}_w^n, \dot{z}_j)$ on the right (or vice versa) and an $[r]$ -move was played at this node then Proposition 7 tells us that \dot{z}_j and \dot{z}_j are not r -connected. Therefore, $r \in \{c, d\}$ and, again, there are not more than three possibilities for the successor node η_{l+1} .
 1. η_{l+1} contains $(\mathcal{H}_w^n, \dot{z}_j)$ on the left and $(\mathcal{H}_w^n, \dot{z}_j)$ on the right (or vice versa) and these models are marked;
 2. η_{l+1} contains $(\mathcal{H}_w^n, \dot{y}_j)$ on the left and $(\mathcal{H}_w^n, \dot{y}_j)$ on the right (or vice versa) and these models are marked;
 3. η_{l+1} contains $(\mathcal{H}_w^n, \dot{x}_{j-1})$ on the left and $(\mathcal{H}_w^n, \dot{x}_{j-1})$ on the right (or vice versa) and these models are marked.

It is obvious that every node of B contains a marked pair of one of the following forms

- $(\mathcal{H}_w^n, \dot{x}_j)$ on the left and $(\mathcal{H}_w^n, \dot{x}_j)$ on the right or vice versa;
- $(\mathcal{H}_w^n, \dot{z}_j)$ on the left and $(\mathcal{H}_w^n, \dot{z}_j)$ on the right or vice versa;
- $(\mathcal{H}_w^n, \dot{y}_j)$ on the left and $(\mathcal{H}_w^n, \dot{y}_j)$ on the right or vice versa.

Given the construction of the models \mathcal{H}_w^n , we see that no node of T can be closed if it contains a marked pair that is different from the pair $(\mathcal{H}_w^n, \dot{y}_1)$ on the left and $(\mathcal{H}_w^n, \dot{y}_1)$ on the right. It is obvious that B ends with a node of this form. The reader can easily check that $I(B)$ is an ℓ -path over w in the model $(\mathcal{H}_w^n, \dot{x}_n)$. This follows from the fact that every node labelled with $[i]$ contains a marked model of one of the forms $(\mathcal{H}_w^n, \dot{x}_j)$ or $(\mathcal{H}_w^n, \dot{z}_j)$ or $(\mathcal{H}_w^n, \dot{y}_j)$. \square

Since for every $n > 0$, we have that both $\mathcal{W}_{\Gamma(\text{cd}\Gamma)^*}^n$ and $\mathcal{W}_{(\text{cd}\Gamma)^*}^n$ contain 2^n different words, the next lemma is crucial for the proof of Theorem 4.

Lemma 6. For any two words $w, \underline{w} \in \mathcal{W}_{\Gamma(\text{cd}\Gamma)^*}^n$ or $w, \underline{w} \in \mathcal{W}_{(\text{cd}\Gamma)^*}^n$ the following holds.

For any two pointed models $(\mathcal{A}_w^n, \dot{x}_n)$ and $(\mathcal{A}_{\underline{w}}^n, \dot{x}_n)$, if $w \neq \underline{w}$, then for any two ℓ -paths π over w and $\underline{\pi}$ over \underline{w} , it is true that $\pi \neq \underline{\pi}$. Similarly, for any two pointed models $(\mathcal{C}_w^n, \dot{x}_n)$ and $(\mathcal{C}_{\underline{w}}^n, \dot{x}_n)$, and also, similarly for any two pointed models $(\mathcal{E}_w^n, \dot{x}_n)$ and $(\mathcal{E}_{\underline{w}}^n, \dot{x}_n)$.

The proof of Lemma 6 follows the steps below.

Step 1. We begin with the observation that for any $w \in \mathcal{W}_{\Gamma(\text{cd}\Gamma)^*}^n \cup \mathcal{W}_{(\text{cd}\Gamma)^*}^n$, it is true that w does not contain any sub-word of the form xx or xyx , where $x, y \in \{a, b, c, d\}$. Therefore, the canonical ℓ -path π_w does not contain two successive indices π_k, π_{k+1} such that $\pi_k = \pi_{k+1}$ or three successive indices $\pi_{k-2}, \pi_{k-1}, \pi_k$, such that $\pi_{k-2} = \pi_k$.

We now argue that π_w is the only ℓ -path over w with this property: Suppose $\underline{\pi} \neq \pi_w$ satisfies the same properties: it gives rise to some extended ℓ -path $u_n = \dot{p}_1 \underline{\pi}_1 \dot{p}_2, \dots, \dot{p}_k \underline{\pi}_k \dot{p}_{m+1}$. We know that π_w is the shortest ℓ -path over w , so, since $\underline{\pi}$ is different, there is a \dot{p}_i such that either $\dot{p}_{i+1} = \dot{p}_i$, or else $\dot{p}_{i+2} = \dot{p}_i$. Consider the first \dot{p}_i in the sequence for which this is the case. We consider two cases.

(1) If \dot{p}_1 is \dot{x}_n , the extended path starts as either $\dot{x}_n \underline{\pi}_1 \dot{x}_n$ or as $\dot{x}_n \underline{\pi}_1 \dot{y}_n \underline{\pi}_2 \dot{x}_n$. Since an ℓ -path only allows one step in \dot{x}_n , in the first case the extended path will start like $\dot{x}_n \underline{\pi}_1 \dot{x}_n \underline{\pi}_1$ (and hence the path starts as $\underline{\pi}_1 \underline{\pi}_1$), or as $\dot{x}_n \underline{\pi}_1 \dot{y}_n \underline{\pi}_2 \dot{x}_n \underline{\pi}_1$ (in which case the path starts as $\underline{\pi}_1 \underline{\pi}_2 \underline{\pi}_1$). In other words, $\underline{\pi}$ does not satisfy the properties we assumed.

(2) The second case is that $\dot{p} \neq \dot{\lambda}_n$. In that case, the extended path starts as either $\dot{p}_1\pi_1\dot{p}_2 \dots \dot{p}_{i-1}\pi_{i-1}\dot{p}_i\pi_i\dot{p}_i$ or as $\dot{p}_1\pi_1\dot{p}_2 \dots \dot{p}_{i-1}\pi_{i-1}\dot{p}_i\pi_i\dot{p}_{i+1}\pi_{i+1}\dot{p}_i$, where all of $\dot{p}_1, \dot{p}_2, \dots, \dot{p}_i$ are different. This implies that the index π_{i-1} represents a step ‘up’ in the left support: if $\dot{p}_i = \dot{\lambda}_j$ then $\pi_{i-1} = d$, if $\dot{p}_i = \dot{\lambda}_j$ then π_{i-1} is either a or b (depending on w), and if $\dot{p}_i = \dot{\lambda}_j$ then $\pi_{i-1} = c$. Let us assume $\dot{p}_i = \dot{\lambda}_j$ for some j (the other cases are similar). This means that $\pi_{i-1} = c$. Also, note that the only allowed steps from $\dot{\lambda}_j$ in an ℓ -path are by using c or d .

(2a) In the case that $\dot{p}_{i+1} = \dot{p}_i = \dot{\lambda}_j$, the sequence $\pi_{i-1}\dot{p}_i\pi_i\dot{p}_{i+1}\pi_{i+1}$ either satisfies $\pi_i = c$ (in which case π contains a string cc), or $\pi_i = d$ and $\pi_{i+1} \in \{c, d\}$, in which case π either contains the string cdc or the string cdd . Again, it was assumed that π did not contain such a string.

(2b) If $\dot{p}_{i+2} = \dot{p}_i$, we obtain the sequence $\pi_{i-1}\dot{p}_i\pi_i\dot{p}_{i+1}\pi_{i+1}\dot{p}_{i+2}$ in the extended ℓ -path. We may assume that $\dot{p}_{i+1} \neq \dot{p}_i$ (since this was dealt with under (2a)). Since we assume $\dot{p}_i = \dot{\lambda}_j$, and that \dot{p}_i is the first in the extended path that is visited twice, we have that the sequence $\pi_{i-1}\dot{p}_i\pi_i\dot{p}_{i+1}\pi_{i+1}\dot{p}_{i+2}$ must be equal to $c\dot{\lambda}_jd\dot{\lambda}_{j-1}d\dot{\lambda}_j$. This means that π contains the substring dd , which we assumed it not to have.

Step 2. We formulate a rewriting rule as follows.

Let an ℓ -path $\pi = \pi_1\pi_2 \dots \pi_n$ over a word w be given. Reading π from left to right, if a substring of the form xx or xyx is encountered, we replace it with x , and continue with the symbols following xx or xyx (if any). Having reached the end of π , we go back to the leftmost symbol of the newly obtained word and repeat the procedure. This algorithm terminates if no substring of the form xx or xyx is encountered.

It is obvious that this algorithm always terminates; moreover it has the following important properties.

1. If π is an ℓ -path over w , then replacing a substring of the form xx or xyx with x in π results in a new ℓ -path π_1 over w ;
2. The procedure terminates with an ℓ -path π^* over w that does not contain any substring of the form xx or xyx .

Therefore, using Step 1, we see that $\pi^* = \pi_w$.

To see item 1, we use the cases of the proof of Lemma 4. Take an ℓ -path $\pi = \pi_1\pi_2 \dots \pi_k$ over w , and let $\varepsilon = \dot{p}_1\pi_1\dot{p}_2\pi_2 \dots \pi_k\dot{p}_{k+1}$ be an extended ℓ -path for π .

1. Suppose $\pi_m = \pi_{m+1} = x$. Define an extended ℓ -path ε' as follows:
 - (a) if ε contains a string $e = \dot{p}_m x \dot{p}_m$, then replace e with \dot{p}_m and call the result ε' ;
 - (b) if ε contains $e = \dot{p}_m x \dot{p}_{m+1} x \dot{p}_m$ with $\dot{p}_m \neq \dot{p}_{m+1}$, replace e with $\dot{p}_m x \dot{p}_m$ and call the result ε' ;
 - (c) if ε contains $e = \dot{p}_m x \dot{p}_{m+1} x \dot{p}_{m+1}$ with $\dot{p}_m \neq \dot{p}_{m+1}$, replace e with $\dot{p}_m x \dot{p}_{m+1}$ and call the result ε' .
2. Suppose $\pi_m \pi_{m+1} \pi_{m+2} = xyx$ with $x \neq y$. Define an extended ℓ -path ε as follows:
 - (a) if ε contains $\dot{p}_m x \dot{p}_m$ suppose first that ε contains the substring $e = \dot{p}_m x \dot{p}_m y \dot{p}_m x \dot{p}_{m+3}$. In this case, replace e with $\dot{p}_m x \dot{p}_{m+3}$ and call the result ε' . This is the only case to consider here, since if ε would have a substring $e = \dot{p}_m x \dot{p}_m y \dot{p}_{m+2}$ with $\dot{p}_m \neq \dot{p}_{m+2}$, then, given the definition of ℓ -path and the definition of words w , it is impossible to extend e with an x -step (it would take a step that is also horizontally possible);
 - (b) if ε contains $e = \dot{p}_m x \dot{p}_{m+1} y \dot{p}_m$ with $\dot{p}_m \neq \dot{p}_{m+1}$ replace e with \dot{p}_m and call the result ε' ;
 - (c) finally, suppose ε contains $e = \dot{p}_m x \dot{p}_{m+1} y \dot{p}_{m+1}$ with $\dot{p}_m \neq \dot{p}_{m+1}$. Since e is to be followed by an x , it must be that the substring in ε is of the form $f = \dot{p}_m x \dot{p}_{m+1} y \dot{p}_{m+1} x \dot{p}_m$. Obviously, we can replace f in ε with $\dot{p}_m x \dot{p}_m$ and call the result ε' .

It should be clear that the extended path ε' thus obtained is an extended ℓ -path, and the ℓ -path π' that ε' induces is exactly π with either xx replaced by x (in case (1)) or xyx replaced by x (in case of (2)).

Step 3. Suppose that there are two pointed models $(\mathcal{H}_w^n, \dot{\lambda}_n)$ and $(\mathcal{H}_{w'}^n, \dot{\lambda}_n)$ such that there are two ℓ -paths π over w and $\underline{\pi}$ over \underline{w} respectively, for which $\pi = \underline{\pi}$. We apply the rewriting rule to π and obtain π_w . Since $\underline{\pi}$ is equal to π , we see that $\pi_w = \pi_{\underline{w}}$. Hence $w = \underline{w}$.

Having established these steps, the proof of Theorem 4 now follows directly, which, for the completeness' sake, we formulate once more.

Theorem 5. Let $\Sigma = \langle A, I \rangle$ be a signature such that I contains at least 4 indices $\{a, b, c, d\}$ and A contains at least 3 propositional symbols $\{a, b, c\}$. For any logic $L \in \{\forall_I\text{ML}, \exists_I\text{ML}, [\varphi]\text{ML}\}$ in the signature Σ , it is true that $L \not\leq_{S5}^{\text{SUBEXP}} \text{ML}$.

Proof. In each case, we use Lemma 1 as a sufficient condition for proving succinctness. The formulae α_n in this lemma are defined in Table 3 by ϑ_n in case of $L = [\forall_I]\text{ML}$, as σ_n in case of $[\exists_I]\text{ML}$, and as φ_n in case of $[\varphi]\text{ML}$. From Propositions 4 and 6, we know that for each of those formulae, and for each n , there are classes of pointed models \mathbb{H}_1^n and \mathbb{H}_2^n such that each $(\mathcal{H}_w^n, \dot{\lambda}_n) \in \mathbb{H}_1^n$ satisfies the formula, while each $(\mathcal{H}_w^n, \dot{\lambda}_n) \in \mathbb{H}_2^n$ falsifies it, where w is a word over a given alphabet. By Lemma 5 we know that every tree $T \in \mathcal{T}((\mathcal{H}_w^n, \dot{\lambda}_n) \circ (\mathcal{H}_{w'}^n, \dot{\lambda}_n))$ has a branch B such that $I(B)$ is an ℓ -path over w in $(\mathcal{H}_w^n, \dot{\lambda}_n)$. Since for every two words w and w' , the ℓ -paths over them are different, and for every n we have 2^n many words, we can apply Theorem 2, the principle of diverging pairs, to all trees $T_1 \in \mathcal{T}((\mathcal{H}_{w_1}^n, \dot{\lambda}_n) \circ (\mathcal{H}_{w_1}^n, \dot{\lambda}_n)), \dots, T_{2^n} \in \mathcal{T}((\mathcal{H}_{w_{2^n}}^n, \dot{\lambda}_n) \circ (\mathcal{H}_{w_{2^n}}^n, \dot{\lambda}_n))$, to conclude that any game tree $T \in \mathcal{T}((\mathbb{H}_1^n \circ \mathbb{H}_2^n))$ has at least 2^n different branches. This implies that in order to win the game starting at $(\mathbb{H}_1^n \circ \mathbb{H}_2^n)$, Spoiler needs at least 2^n moves, so, by Theorem 1, every formula β_n in the logic ML equivalent to the given α_n has length at least 2^n . \square

4. Conclusion

We presented several succinctness results on three extensions of multimodal logic. Clearly, we left some open questions which we think are worthwhile studying. In particular, we do not know whether our bounds on the number of relations and propositional symbols are optimal or not, i.e., to prove that the logic $[\varphi]\text{ML}$ is exponentially more succinct than ML on **S5**, we needed a signature with at least 4 relation indices and 3 propositional variables, whereas the same result for **K** was achieved with only two relation indices and one propositional letter. Similarly, we needed a signature with 4 relation indices and one propositional variable to prove that $[\forall_I]\text{ML}$ and $[\exists_I]\text{ML}$ are exponentially more succinct than ML on **S5**, whereas this could be done with only two indices and one propositional variable in the **K** case.

It was proven in [24] that $[\forall_I]\text{ML}$ is exponentially more succinct than $[\exists_I]\text{ML}$ (and vice versa) on **K**. We conjecture that a similar succinctness result can be obtained with respect to **S5**. Similarly, it was shown in [15] that $[\varphi]\text{ML}$ is exponentially more succinct than both $[\exists_I]\text{ML}$ and $[\forall_I]\text{ML}$ on **K**. Again, we conjecture that the same is true on **S5**, too. We do not know whether one of $[\exists_I]\text{ML}$ or $[\forall_I]\text{ML}$ is exponentially more succinct than $[\varphi]\text{ML}$ either on **K** or **S5**.

On a more general note, some of the currently known succinctness gaps between different logics are conditional on certain assumptions (be it rather common ones) on computational complexity, e.g., [4,10,12,17]. It remains to be seen whether Adler–Immerman games or other techniques can be used to eliminate the use of such conjectures on computational complexity in the proofs of the results mentioned above.

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